

Stability of periodic travelling shallow-water waves determined by Newton's equation

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Abstract. We study the existence and stability of periodic travelling-wave solutions for generalized Benjamin-Bona-Mahony and Camassa-Holm equations. To prove orbital stability, we use the abstract results of Grillakis-Shatah-Strauss and the Floquet theory for periodic eigenvalue problems.

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1. Introduction.

Consider the following equation

$$u_t + (a(u))_x - u_{xxt} = \left(b'(u) \frac{u_x^2}{2} + b(u) u_{xx} \right)_x \quad (1.1)$$

where $a, b : \mathbb{R} \rightarrow \mathbb{R}$ are smooth functions and $a(0) = 0$. In this paper we study the problems of the existence and stability of periodic travelling-wave solutions $u = \varphi(x - vt)$ for (1.1). It is easy to see that whatever a, b be, the equation for φ has no dissipative terms. Hence, any travelling-wave solution of (1.1) is determined from Newton's equation which we will write below in the form $\varphi'^2 = U(\varphi)$. Therefore by using the well-known properties of the phase portrait of Newton's equation in the (φ, φ') -plane, one can establish that under fairly broad conditions, (1.1) has at least one three-parameter family of periodic solutions $\varphi(y) = \varphi(v, c_1; \varphi_0; y)$ where c_1 is a constant of integration and $\varphi_0 = \min \varphi$ (see Proposition 1). The parameters v, c_1 determine the phase portrait whilst φ_0 serves to fix the periodic orbit within. Moreover, if $T = T(v, c_1, \varphi_0)$ is the minimal (sometimes called *fundamental*) period of φ , then φ has exactly one local minimum and one local maximum in $[0, T)$. Therefore φ' has just two zeroes in each semi-open interval of length T . By Floquet theory, this means that φ' is either the second or the third eigenfunction of the periodic eigenvalue problem obtained from the second variation along φ of an appropriate conservative functional $M(u)$. If the first case occurs, then one can use the abstract result of Grillakis-Shatah-Strauss ([25]) to prove orbital stability whenever $\ddot{d}(v) = (d^2/dv^2)M(\varphi)$ is positive.

In the periodic case we deal with, it is not always so easy to determine the sign of $\ddot{d}(v)$. To overcome this problem, we first establish a general result (see Proposition

6) expressing $\ddot{d}(v)$ through some special line integrals along the energy level orbit $\{H = h\}$ of the Newtonian function $H(X, Y) = Y^2 - U(X)$ which corresponds to φ . When a, b are polynomials, these are complete Abelian integrals and one can apply methods from algebraic geometry (Picard-Fuchs equations, etc.) to determine the possible values of v , c_1 and φ_0 where $\ddot{d}(v)$ changes sign. Let us mention that even for v and c_1 fixed, the sign of $\ddot{d}(v)$ might depend on the amplitude of φ (ruled by φ_0) as shown in Proposition 8. In this connection, we calculate explicitly the main term of $\ddot{d}(v)$ in the case of arbitrary small-amplitude periodic solutions φ of (1.1), see formula (7.7). It is shown that the main term depends on the first two isochronous constants related to the center $(X_0, 0)$ into which the orbit (φ, φ') shrinks when $\varepsilon = \max \varphi - \min \varphi \rightarrow 0$, and on X_0 itself as well.

We apply our results to prove orbital stability for several particular examples.

Theorem I. (The modified BBM equation). *Let $a(u) = 2\omega u + \beta u^3$, $b(u) = 0$, $\beta > 0$ and $u = \varphi(x - vt)$ where $v > 0$, $\varphi(y) = \varphi(v, 0; \varphi_0; y)$ be a periodic travelling-wave solution of (1.1) which does not oscillate around zero. Then φ is orbitally stable in any of the cases:*

- (i) $3v^2 - 8\omega^2 \geq 0$;
- (ii) $3v^2 - 8\omega^2 < 0$, $2v^2 - 2\omega v - \omega^2 > 0$ and the period of φ is sufficiently large.

Theorem II. (The perturbed single-power BBM equation). *Let $a(u) = \beta u^2$, $b(u) = \gamma \beta u$, $\beta > 0$, and let $u = \varphi(x - vt)$ where $v > 0$, $\varphi(y) = \varphi(v, 0; \varphi_0; y)$ be a periodic travelling-wave solution of (1.1). Then φ is orbitally stable for small $|\gamma|$.*

Theorem III. (The perturbed single-power mBBM equation). *Let $a(u) = \beta u^3$, $b(u) = \gamma \beta u^2$, $\beta > 0$, and let $u = \varphi(x - vt)$ where $v > 0$, $\varphi(y) = \varphi(v, 0; \varphi_0; y)$ be a periodic travelling-wave solution of (1.1) which does not oscillate around zero. Then φ is orbitally stable for small $|\gamma|$.*

Theorem IV. (Small-amplitude waves of the perturbed BBM equation.) *Let $a(u) = 2\omega u + \frac{3}{2}u^2$, $b(u) = \gamma g(u)$ and $u = \varphi(x - vt)$ where $v > 0$, $\varphi(y) = \varphi(v, c_1; \varphi_0; y)$ be a periodic travelling-wave solution of (1.1) having a small amplitude. Then φ is orbitally stable for small $|\gamma|$ and $(\omega/v, c_1/v^2)$ taken in appropriate domain $\Omega \subset \mathbb{R}^2$.*

Theorem V. (Small-amplitude waves of the perturbed mBBM equation). *Let $a(u) = 2\omega u + \beta u^3$, $b(u) = \gamma g(u)$, $\beta > 0$ and $u = \varphi(x - vt)$ where $v > 0$, $\varphi(y) = \varphi(v, 0; \varphi_0; y)$ be a periodic travelling-wave solution of (1.1) which has a small amplitude and does not oscillate around zero. Then φ is orbitally stable for $3v^2 - 8\omega^2 > 0$ and small $|\gamma|$.*

We point out that, unlike the other cases, in Theorem IV the constant of integration c_1 is not fixed, therefore we consider the whole family of small-amplitude waves. The explicit expression of Ω is given in the proof.

Let us mention that for $a(u) = 2ku + \frac{3}{2}u^2$ and $b(u) = u$, equation (1.1) becomes

the well-known Camassa-Holm equation

$$u_t + 2ku_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}. \quad (1.2)$$

Equation (1.2) was derived as a bi-Hamiltonian generalization of the Korteweg-de Vries equation [23] and later Camassa and Holm [10] recovered it as a water-wave model. The Camassa-Holm equation is locally well-posed in H^s for $s > \frac{3}{2}$. Moreover, while some solutions of equation (1.2) are global, others blow up in finite time (in both the periodic and non-periodic cases) [11, 12, 13, 14, 15, 38, 42]. The solitary waves of Camassa-Holm equation are smooth in the case $k > 0$ and peaked for $k = 0$. Their stability is considered in [16, 17, 18, 26, 27, 36, 37].

For $a(u) = 2ku + \frac{3}{2}u^2$ and $b(u) = \gamma u$, equation (1.1) serves as a model equation for mechanical vibrations in a compressible elastic rod [21, 22]. Some problems such as well-posedness and blowing-up in this case were studied in [41, 42, 44], and stability of solitary waves was investigated in [19, 33].

If $b(u) = 0$ in (1.1), one obtains the generalized Benjamin-Bona-Mahony (gBBM) equation for surface waves in a channel [8]. All solutions are global and their solitary waves are stable or unstable depending on $a(u)$ [30, 40].

For equation (1.1), the well-posedness and stability of solitary waves in the case $a(u) = 2ku + \frac{p+2}{2}u^{p+1}$ and $b(u) = u^p$ are studied in [26]. For $k = 0$, equation (1.1) admits peaked solitary wave solutions, which are stable (see [27]).

The existence and stability of periodic travelling waves for nonlinear evolution equations has received little attention. Recently Angulo, Bona and Scialom in [4] developed a complete theory of the stability of cnoidal waves for KdV equation. The solution $u(x, t) = \varphi_c(x - ct)$ of KdV satisfies the equation

$$\varphi_c'' + \frac{1}{2}\varphi_c^2 - c\varphi_c = A_{\varphi_c},$$

where A_{φ_c} is an integration constant. An explicit form for φ_c in the periodic case is

$$\varphi_c(\xi) = \beta_2 + (\beta_3 - \beta_2)cn^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12}}\xi; k\right),$$

where cn is the Jacobi elliptic function and the following relations take place:

$$\beta_1 < \beta_2 < \beta_3, \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}, \quad \beta_1 + \beta_2 + \beta_3 = 3c, \quad A_{\varphi_c} = -\frac{1}{6}\sum_{i < j} \beta_i \beta_j.$$

Since $cn(u + 2K) = -cn(u)$ where $K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ is the complete elliptic integral of the first kind, then φ_c has the fundamental (i.e. minimal) period T_{φ_c} given by

$$T_{\varphi_c} = \frac{4\sqrt{3}}{\sqrt{\beta_3 - \beta_1}}K(k).$$

Moreover, T_{φ_c} depends on the speed c and satisfies the inequality

$$T_{\varphi_c}^2 > \frac{(2\pi)^2}{\sqrt{c^2 + 2A_{\varphi_c}}}$$

Using the abstract results of Grillakis, Shatah and Strauss (adapted to the periodic context), the authors proved in [4] that the cnoidal waves with mean value zero are orbitally stable.

Other new explicit formulas for the periodic travelling waves based on the Jacobi elliptic function of type dnoidal, together with their stability, have been obtained by Angulo [2, 3] for the nonlinear Schrödinger (NLS) equation $iu_t + u_{xx} + |u|^2u = 0$, modified KdV equation $u_t + 3u^2u_x + u_{xxx} = 0$ and Hirota-Satsuma system

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) = 2bv v_x \\ v_t + v_{xxx} + 3uv_x = 0. \end{cases}$$

For NLS the solutions are of the form $u(x, t) = e^{i\omega t}\varphi_w(x)$ where $\varphi_w(x)$ is a periodic smooth function with period $L > 0$. The solution φ_w is of the form

$$\varphi_w(x) = \eta_1 \operatorname{dn} \left(\frac{\eta_1}{\sqrt{c}} x; k \right)$$

where η_1 and the modulus k depend smoothly on w . The orbit Ω_{φ_w} ,

$$\Omega_{\varphi_w} = \{e^{i\theta}\varphi_w(\cdot + y), (y, \theta) \in \mathbb{R} \times [0, 2\pi)\}$$

generated by the dnoidal wave φ_w is stable by perturbation of periodic function with period L and nonlinearly unstable by perturbation of periodic function with period $2L$. In all these works it was necessary to use an elaborated spectral theory for the periodic eigenvalue problem,

$$\begin{cases} \frac{d^2}{dx^2}\Psi + [\rho - n(n+1)k^2 \operatorname{sn}^2(x; k)]\Psi = 0, \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases} \quad (1.3)$$

with specific values of $n \in \mathbb{N}$. We will also make use of systems similar to (1.3).

The present paper is organized as follows. In section 2 we formulate and sketch the proof of a local well-posedness result for the equation (1.1) in periodic H^s spaces. In Section 3 we prove the existence of periodic travelling waves of a given (admissible) period and study their properties. In Section 4 we prove the orbital stability result under some hypotheses (see Assumption 1). In Section 5 several particular examples are considered. For most of them, we determine φ explicitly and show that φ' is the second eigenfunction of the respective periodic eigenvalue problem. We also determine the sign of $\ddot{d}(v)$ to outline the cases satisfying Assumption 1. In Section 6 perturbation theory is applied to prove the orbital stability of periodic travelling waves for generalized Benjamin-Bona-Mahony and generalized Camassa-Holm equations (the case of small b in the right-hand side of (1.1)). In Section 7 we study the small-amplitude periodic travelling waves of (1.1) and determine the sign of $\ddot{d}(v)$ for them.

2. Local well-posedness.

In this section, we discuss the local well-posedness of the Cauchy problem for equation (1.1). We begin by introducing some notation and by recalling related definitions we shall use throughout the paper.

Let $\mathcal{P} = C_{per}^\infty$ denote the collection of all functions which are C^∞ and periodic with a period $T > 0$. The topological dual of \mathcal{P} will be denoted by \mathcal{P}' . If $\Psi \in \mathcal{P}'$ then we denote by $\Psi(f) = \langle \Psi, f \rangle$ the value of Ψ at f . Define the functions $\Theta_k(x) = \exp(2\pi i k x / T)$, $k \in \mathbb{Z}$. The Fourier transform of $\Psi \in \mathcal{P}'$ is the function $\hat{\Psi} : \mathbb{Z} \rightarrow \mathbb{C}$ defined by $\hat{\Psi}(k) = \frac{1}{T} \langle \Psi, \Theta_{-k} \rangle$. If Ψ is a periodic function with a period T , we have

$$\hat{\Psi}(k) = \frac{1}{T} \int_0^T \Psi(x) \exp(-2\pi i k x / T) dx.$$

For $s \in \mathbb{R}$, the Sobolev space $H^s([0, T])$ is the set of all $f \in \mathcal{P}'$ such that

$$\|f\|_s^2 = T \sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 < \infty.$$

Certainly, $H^s([0, T])$ is a Hilbert space with respect to the inner product

$$(f, g)_s = T \sum_{k=-\infty}^{\infty} (1 + |k|^2)^s \hat{f}(k) \overline{\hat{g}(k)}.$$

Since $H^s([0, T]) \subset L^2([0, T])$ for every $s \geq 0$, we obtain via Plancherel identity that for every $n \in \mathbb{N}$,

$$\|f\|_n^2 = \sum_{j=0}^n \|f^{(j)}\|^2.$$

where $f^{(j)}$ represents the j th derivative of f taken in the sense of \mathcal{P}' . Moreover, Sobolev's lemma states that if $s > l + \frac{1}{2}$, then $H^2([0, T]) \hookrightarrow C_{per}^l$ where

$$C_{per}^l = \{f \in C^l : f^{(j)} \text{ is periodic with a period } T \text{ for } j = 0, \dots, l\}.$$

One can prove the following result about equation (1.1).

Theorem 1. *Assume that $a, b \in C^{m+3}(\mathbb{R})$, $m \geq 2$. Given $u_0 \in H^s$, $\frac{3}{2} < s < m$, there exists a maximal $t_0 > 0$ and a unique solution $u(x, t)$ to (1.1) such that*

$$u \in C([0, t_0], H^s) \cap C^1([0, t_0], H^{s-1}).$$

Moreover, the solution depends continuously on the initial data.

Proof. Take $u \in H^s$ and let

$$A(u) = b(u) \partial_x, \quad f(u) = (1 - \partial_x^2)^{-1} [b(u) u_x - \partial_x (\frac{1}{2} b'(u) u_x^2 + a(u))].$$

Using the above notations, one can rewrite equation (1.1) in the following form:

$$u_t + A(u)u = f(u).$$

In a similar way as in Theorem 2.2 in [27] (dealing with the non-periodic case), we have

(1) $A(u)$ is quasi-m-accretive, uniformly on the bounded sets in H^{s-1} . Moreover, $A(u) \in L(H^s, H^{s-1})$ (where $L(X, Y)$ is the space of all linear bounded operators from X to Y , $L(X) = L(X, X)$) and

$$\|(A(u_1) - A(u_2))u_3\|_{s-1} \leq \mu_1 \|u_1 - u_2\|_{s-1} \|u_3\|_s.$$

(2) Define $\Lambda = (1 - \partial_x^2)^{1/2}$, $B(u) = [\Lambda, b(u)\partial_x]\Lambda^{-1}$ for $u \in H^s$, where $[\Lambda, M]$ denotes the commutator of Λ and M . Then $B(u) \in L(H^{s-1})$ and

$$\|(B(u_1) - B(u_2))u_3\|_{s-1} \leq \mu_2 \|u_1 - u_2\|_s \|u_3\|_{s-1}, \quad u_1, u_2 \in H^s, \quad u_3 \in H^{s-1}.$$

(3) $f(u)$ is bounded on the bounded sets in H^s and satisfies

$$\|f(u_1) - f(u_2)\|_s \leq \mu_3 \|u_1 - u_2\|_s, \quad u_1, u_2 \in H^s,$$

$$\|f(u_1) - f(u_2)\|_{s-1} \leq \mu_4 \|u_1 - u_2\|_{s-1}, \quad u_1, u_2 \in H^{s-1}.$$

Applying Kato's theory for abstract quasilinear evolution equations [35], we obtain the local well-posedness of the equation (1.1) in H^s , for $\frac{3}{2} < s < m$. The solution $u(x, t)$ belongs to $C([0, t_0], H^s) \cap C^1([0, t_0], H^{s-1})$.

3. Periodic travelling-wave solutions.

We are looking for a travelling-wave solution of (1.1) of the form $u(x, t) = \varphi(x - vt)$. We assume that φ is smooth and bounded in \mathbb{R} . The following two cases appear:

- (i) $\varphi' \neq 0$ in \mathbb{R} and $\varphi_- < \varphi < \varphi_+$ (corresponding to kink-wave solution);
- (ii) $\varphi'(\xi) = 0$ for some $\xi \in \mathbb{R}$. Denote $\varphi_0 = \varphi(\xi)$, $\varphi_2 = \varphi''(\xi)$.

Below we will deal with the second case. Replacing in (1.1) we get

$$-v\varphi' + (a(\varphi))' + v\varphi''' = \left(b'(\varphi)\frac{\varphi'^2}{2} + b(\varphi)\varphi'' \right)'. \quad (3.1)$$

By integrating (3.1) twice, one obtains

$$-v\varphi + a(\varphi) + v\varphi'' = b'(\varphi)\frac{\varphi'^2}{2} + b(\varphi)\varphi'' + c_1, \quad (3.2)$$

$$[v - b(\varphi)]\frac{\varphi'^2}{2} = c_2 + c_1\varphi + \frac{v}{2}\varphi^2 - A(\varphi), \quad A(\varphi) = \int_0^\varphi a(s)ds, \quad (3.3)$$

with some constants c_1, c_2 . In case (ii), one has respectively

$$\begin{aligned} c_1 &= a(\varphi_0) - v\varphi_0 + (v - b(\varphi_0))\varphi_2, \\ c_2 &= A(\varphi_0) + \frac{1}{2}v\varphi_0^2 - \varphi_0a(\varphi_0) - (v - b(\varphi_0))\varphi_0\varphi_2 = A(\varphi_0) - \frac{1}{2}v\varphi_0^2 - c_1\varphi_0. \end{aligned}$$

Next we are going to look for periodic travelling-wave solutions φ . Consider in the plane $(X, Y) = (\varphi, \varphi')$ the Hamiltonian system

$$\begin{aligned}\dot{X} &= [v - b(X)]Y = H_Y, \\ \dot{Y} &= \frac{1}{2}b'(X)Y^2 - a(X) + vX + c_1 = -H_X,\end{aligned}\tag{3.4}$$

with a Hamiltonian function

$$H(X, Y) = [v - b(X)]\frac{Y^2}{2} + A(X) - \frac{v}{2}X^2 - c_1X.$$

Then (3.3) becomes $H(\varphi, \varphi') = c_2$ and the curve $s \rightarrow (\varphi(s-s_0), \varphi'(s-s_0))$ determined by (3.3) lies on the energy level $H = c_2$ of the Hamiltonian $H(X, Y)$. Within the analytical class, system (3.4) has periodic solutions if and only if it has a center. Each center is surrounded by a continuous band of periodic trajectories (called *period annulus*) which terminates at a certain separatrix contour on the Poincaré sphere. As far as the straight lines $X = X_0$ where $v - b(X_0) = 0$ are (unions of) trajectories, the critical points of center type of (3.4) are given by the critical points on $Y = 0$ having a negative Hessian. These are the points $(X_0, 0)$ where:

$$c_1 + vX_0 - a(X_0) = 0, \quad [v - b(X_0)][v - a'(X_0)] < 0.\tag{3.5}$$

(For simplicity, we will not consider here the case of a degenerate center when the Hessian becomes zero.)

The above considerations lead us to the following statement.

Proposition 1. *Let c_1 and v be constants such that conditions (3.5) are satisfied for some $X_0 \in \mathbb{R}$. Then there is an open interval Δ containing X_0 such that:*

(i) *For any $\varphi_0 \in \Delta$, $\varphi_0 < X_0$, the solution of (1.2) satisfying*

$$\varphi(\xi) = \varphi_0, \quad \varphi'(\xi) = 0, \quad \varphi''(\xi) = \frac{c_1 + v\varphi_0 - a(\varphi_0)}{v - b(\varphi_0)},$$

is periodic.

(ii) *If $\varphi_1 \in \Delta$, $\varphi_1 > X_0$ is the nearest to X_0 solution of $H(X, 0) = H(\varphi_0, 0)$, then $\varphi_0 \leq \varphi \leq \varphi_1$.*

(iii) *If T is the minimal period of φ , then in each interval $[s, s + T)$ the function φ has just one minimum and one maximum (equal to φ_0 and φ_1 , respectively) and is strictly monotone elsewhere.*

Proof. Part (i) follows from the analysis already done above. Parts (ii) and (iii) follow from the fact that $H(X, Y)$ is symmetric with respect to Y and because for any fixed \bar{X} the equation $H(\bar{X}, Y) = \text{const}$ has just two solutions $\bar{Y}, -\bar{Y}$.

Example. Assume that a, b are polynomials and $\deg a = 2$, $\deg b \leq 1$. For c_1, v properly chosen, the quadratic equation in (3.5) will have two distinct real roots

$X_1 < X_2$. It is easy to see that if $v - b(X) \neq 0$ in $[X_1, X_2]$, then (3.5) holds for just one of these roots. If $a''b' < 0$ and $v - b(X)$ vanishes at X_1 , then (3.5) holds for X_2 , and vice versa. Finally, if $a''b' < 0$ and $v - b(X)$ has a root within (X_1, X_2) , then both X_1 and X_2 satisfy (3.5). To summarize: the phase portraits of all quadratic systems (3.4) (including the exceptional case $\deg a \leq 1$, $\deg b = 1$) satisfying (3.5), that is having a center, are divided into 7 topologically different classes, see e.g. [28].

Remark 1. Below, we shall denote $\Delta^- = \Delta \cap \{(-\infty, X_0)\}$, $\Delta^+ = \Delta \cap \{(X_0, \infty)\}$. It is possible that (3.3) also has periodic solutions for initial values φ_0 far from X_0 . To study them, it is needed to specify the functions a and b in (1.1).

Let us denote

$$U(s) = \frac{2c_2 + 2c_1s + vs^2 - 2A(s)}{v - b(s)} = \frac{2A(\varphi_0) - v\varphi_0^2 - 2c_1\varphi_0 + 2c_1s + vs^2 - 2A(s)}{v - b(s)}.$$

Then for $\varphi_0 \leq \varphi \leq \varphi_1$ one can rewrite (3.3) as $\varphi'(\sigma) = \sqrt{U(\varphi(\sigma))}$. Integrating the equation along the interval $[\xi, s] \subset [\xi, \xi + T/2]$ yields an implicit formula for the value of $\varphi(s)$:

$$\int_{\varphi_0}^{\varphi(s)} \frac{d\sigma}{\sqrt{U(\sigma)}} = s - \xi, \quad s \in [\xi, \xi + T/2]. \quad (3.6)$$

For $s \in [\xi + T/2, \xi + T]$ one has $\varphi(s) = \varphi(T + 2\xi - s)$. We recall that the period function T of a Hamiltonian flow generated by $H_0 \equiv \frac{1}{2}Y^2 - \frac{1}{2}U(X) = 0$ is determined from

$$T = \int_0^T dt = \oint_{H_0=0} \frac{dX}{Y} = 2 \int_{\varphi_0}^{\varphi_1} \frac{dX}{\sqrt{U(X)}} \quad (3.7)$$

This is in fact the derivative (with respect to the energy level) of the area surrounded by the periodic trajectory through the point $(\varphi_0, 0)$ in the $(X, Y) = (\varphi, \varphi')$ -plane.

Consider the continuous family of periodical travelling-wave solutions $\{u = \varphi(x - vt)\}$ of (1.1) and (3.3) going through the points $(\varphi, \varphi') = (\varphi_0, 0)$ where $\varphi_0 \in \Delta^-$. For any $\varphi_0 \in \Delta^-$, denote by $T = T(\varphi_0)$ the corresponding period. One can see (e.g. by using formula (3.7) above) that the period function $\varphi_0 \rightarrow T(\varphi_0)$ is smooth. To check this, it suffices to perform a change of the variable

$$X = \frac{\varphi_1 - \varphi_0}{2}s + \frac{\varphi_1 + \varphi_0}{2} \quad (3.8)$$

in the integral (3.7) and use that

$$U(\varphi_0) = U(\varphi_1) = 0. \quad (3.9)$$

Also, it is not difficult to verify (see Section 7) that

$$T(\varphi_0) \rightarrow T_0 = 2\pi \sqrt{\frac{v - b(X_0)}{a'(X_0) - v}} \quad \text{as} \quad \varphi_0 \uparrow X_0.$$

Conversely, taking v, c_1 to satisfy the conditions of Proposition 1 and fixing T in a proper interval, one can determine φ_0 and φ_1 as smooth functions of v, c_1 so that the periodic solution φ given by (3.6) will have a period T . The condition for this is the monotonicity of the period.

Definition 1. We say that the period $T = T(\varphi_0)$ is *not critical* provided that $T'(\varphi_0) \neq 0$.

If the period $T(\varphi_0)$ is not critical for any $\varphi_0 \in \Delta_-$, then the period function is strictly monotone along the period annulus and its range is an open interval I having T_0 as an endpoint.

Proposition 2. Let \mathcal{A} be a period annulus of (3.4) which surrounds a nondegenerate center and has a monotone period function. Then for any $T \in I$ there is a unique $\varphi_0 \in \Delta_-$ satisfying $T(\varphi_0) = T$. Its derivative $\dot{\varphi}_0$ with respect to v is determined from

$$\dot{\varphi}_0[c_1 + v\varphi_0 - a(\varphi_0)] \frac{d}{dh} \oint_{H=h} \frac{dx}{y} = \frac{d}{dh} \oint_{H=h} \frac{(x^2 + y^2 - \varphi_0^2)dx}{2y} \quad (3.10)$$

where

$$h = A(\varphi_0) - \frac{v}{2}\varphi_0^2 - c_1\varphi_0. \quad (3.11)$$

Proof. We use the implicit function theorem (IFT) and the Gelfand-Leray form (see [6], Chapter 3). Let I be the range of the period function along \mathcal{A} . For $T \in I$ and $\varphi_0 \in \Delta_-$ denote

$$G(v, c_1, \varphi_0) = \bar{G}(v, c_1, h) = T(\varphi_0) - T = \oint_{H=h} \frac{dx}{y} - T.$$

Then

$$0 \neq T'(\varphi_0) = \frac{\partial G}{\partial \varphi_0} = \frac{dh}{d\varphi_0} \frac{\partial \bar{G}}{\partial h} = [a(\varphi_0) - v\varphi_0 - c_1] \frac{d}{dh} \oint_{H=h} \frac{dx}{y}$$

and the IFT works. Hence

$$\bar{G} = \frac{\partial \bar{G}}{\partial h} \dot{h} + \frac{\partial \bar{G}}{\partial v} = 0. \quad (3.12)$$

From $H(x, y) = h$ we obtain the covariant derivatives

$$(v - b(x))y \frac{dy}{dh} = 1, \quad \frac{1}{2}(y^2 - x^2) + (v - b(x))y\dot{y} = 0. \quad (3.13)$$

Therefore, by using Gelfand-Leray form to calculate the derivatives, we get

$$\bar{G}(v, c_1, h) = \frac{d}{dh} \oint_{H=h} (v - b(x))y dx - T, \quad \frac{\partial \bar{G}}{\partial h} = \frac{d}{dh} \oint_{H=h} \frac{dx}{y}, \quad (3.14)$$

$$\frac{\partial \bar{G}}{\partial v} = \frac{d}{dh} \oint_{H=h} [y + (v - b(x))\dot{y}] dx = \frac{d}{dh} \oint_{H=h} \frac{(x^2 + y^2)dx}{2y}. \quad (3.15)$$

Hence, by (3.12), (3.14) and (3.15) we obtain

$$\dot{h} \frac{d}{dh} \oint_{H=h} \frac{dx}{y} + \frac{d}{dh} \oint_{H=h} \frac{(x^2 + y^2)dx}{2y} = 0. \quad (3.16)$$

Finally, from (3.11) one obtains $\dot{h} = -\frac{1}{2}\varphi_0^2 + [a(\varphi_0) - v\varphi_0 - c_1]\dot{\varphi}_0$. Together with (3.16), this implies (3.10). \square

Remark 2. Obviously, one can formulate a local variant of Proposition 2 concerning a given noncritical period $T(\varphi_0)$ only. As far as $T'(\varphi_0) \neq 0$, the same proof clearly goes and no restrictions concerning the period annulus \mathcal{A} are needed.

The perturbation result we establish below will be needed in Section 6. Instead of (1.1), consider now a small perturbation of the generalized BBM equation

$$u_t + (a(u))_x - u_{xxt} = \gamma \left(b'(u) \frac{u_x^2}{2} + b(u)u_{xx} \right)_x, \quad |\gamma| \ll 1 \quad (1.1_\gamma)$$

and let $\{\varphi^\gamma(x - vt)\}$ be the family of corresponding periodic travelling-wave solutions going through points $(\varphi_0^\gamma, 0)$ in the (φ, φ') -plane. Denote the related periods by $T(\varphi_0^\gamma)$.

Proposition 3. *Assume that the period $T = T(\varphi_0^0)$ related to the solution $\varphi^0(x - vt)$ of (1.1₀) is not critical. Then:*

(i) *There is a smooth function $\gamma \rightarrow \varphi_0(\gamma)$ defined for small $|\gamma|$ and satisfying $\varphi_0(0) = \varphi_0^0$, such that the travelling wave solution $\varphi^\gamma(x - vt)$ of (1.1 _{γ}) going through the point $(\varphi_0(\gamma), 0)$ is periodical and has a (minimal) period T .*

(ii) $\max_{[0, T]} |\varphi^\gamma - \varphi^0| = O(\gamma)$.

Proof. (i). The proof relies on the implicit function theorem. Consider system (3.4) with b replaced by γb . Since for $\varphi_0 \in \Delta^-$ the point $(\varphi_0, 0)$ is not critical for (3.4), IFT yields that there is a smooth function $\varphi_1 = \varphi_1(\varphi_0, \gamma)$ determined from $H(\varphi_1, 0) = H(\varphi_0, 0)$, which takes values in Δ^+ and $\varphi_0 \leq \varphi^\gamma \leq \varphi_1$. Introducing a new variable (3.8) in (3.7) and making use of (3.9), we can rewrite (3.7) in the form $G(\gamma, \varphi_0) = 0$ where

$$G(\gamma, \varphi_0) = 2 \int_{-1}^1 \frac{ds}{\sqrt{(1-s^2)U_1(\gamma, \varphi_0, s)}} - T,$$

with U_1 a smooth positive function. We further have

$$G(0, \varphi_0^0) = 0, \quad \left. \frac{dG(\gamma, \varphi_0)}{d\varphi_0} \right|_{\gamma=0} = T'(\varphi_0^0) \neq 0,$$

therefore (i) follows, again by the IFT.

(ii) Consider system (3.4) with b replaced by γb and initial data $X(\xi) = \varphi_0(\gamma)$, $Y(\xi) = 0$. By the uniqueness and the smooth dependence theorems, the solution φ^γ

is smooth and therefore a uniformly Lipschitz continuous function with respect to γ for $|\gamma|$ small, on each interval $[\xi, \xi + T]$. \square

In support to the hypothesis of Proposition 3, we state the following (in fact, known) result about the period $T = T(\varphi_0)$.

Proposition 4. (i) *There is no critical period provided that $a(u)$ in $(1, 1_\gamma)$ is a polynomial of degree 2.* (ii). *There is at most one critical period, if $a(u)$ is a polynomial of degree 3.*

Proof. Consider the period function related to the Hamiltonian system (3.4) (with $b(X) = 0$) as a function $T(h)$ of the energy level $H = h$. It is well known that $T'(h) \neq 0$ if $\deg a = 2$ [20] and that $T'(h)$ has at most one zero if $\deg a = 3$ [24]. As

$$\frac{d}{d\varphi_0} = [a(\varphi_0) - v\varphi_0 - c_1] \frac{d}{dh}$$

and the coefficient is not zero for $\varphi_0 \in \Delta^-$ according to (3.5), the claim follows. \square

4. Conservation laws and conditional stability.

We now turn to our main stability problem. Take $v \in \mathbb{R}$ and denote by φ_v the periodic travelling-wave solution $u = \varphi(x - vt)$ of (1.1).

Definition 2. The periodic travelling-wave solution φ_v of (1.1) is said to be *stable*, if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $u \in C([0, t_0], H^1)$ is a solution of (1.1) satisfying $\|u(0) - \varphi_v\|_1 < \delta$, then

$$\inf_{r \in \mathbb{R}} \|u(\cdot, t) - \varphi(\cdot - r)\|_1 < \varepsilon \quad \text{for every } t \in [0, t_0].$$

For $\varepsilon > 0$, consider a neighborhood \mathcal{U}_ε in the set of all translations of φ_v defined by

$$\mathcal{U}_\varepsilon = \{u \in H^1([0, T]) : \inf_{r \in \mathbb{R}} \|u - \varphi_v(\cdot - r)\|_1 < \varepsilon\}.$$

Lemma 1. *There exists $\varepsilon > 0$ and a C^1 -map $\alpha : \mathcal{U}_\varepsilon \rightarrow \mathbb{R}/T$ such that for all $u \in \mathcal{U}_\varepsilon$, one holds $\langle u(\cdot + \alpha(u)), \varphi'_v \rangle = 0$.*

Proof. Standard (for details, see Lemma 4.1 in [9]).

As usual, an important role in our construction will be played by some functionals which are invariant with respect to time t . Equation (1.1) possesses the following conservation laws:

$$E(u) = - \int_0^T \left[A(u) + \frac{b(u)}{2} u_x^2 \right] dx,$$

$$Q(u) = \frac{1}{2} \int_0^T (u^2 + u_x^2) dx,$$

$$V(u) = \int_0^T u dx$$

where T is the minimal period of the solution $u(x, t)$ and $A'(u) = a(u)$. Let us denote for short $M = E + vQ$. In terms of E , Q , and V equation (3.2) with $\varphi = \varphi_v$ reads

$$M'(\varphi_v) + c_1 = E'(\varphi_v) + vQ'(\varphi_v) + c_1V'(\varphi_v) = 0. \quad (4.1)$$

Let

$$d(v) = M(\varphi_v). \quad (4.2)$$

Then differentiating (4.2) with respect to v we obtain

$$\begin{aligned} \dot{d}(v) &= Q(\varphi_v), \\ \ddot{d}(v) &= \langle Q'(\varphi_v), \dot{\varphi}_v \rangle = \frac{d}{dv} \left(\frac{1}{2} \int_0^T (\varphi_v^2 + \varphi_v'^2) dx \right). \end{aligned} \quad (4.3)$$

Consider in $L^2[0, T]$ the operator \mathcal{H}_v defined by the formal differential expression

$$\mathcal{H}_v = M''(\varphi_v) = (b(\varphi_v) - v)\partial_x^2 + b'(\varphi_v)\varphi_v'\partial_x + v - a'(\varphi_v) + \frac{1}{2}b''(\varphi_v)\varphi_v'^2 + b'(\varphi_v)\varphi_v''. \quad (4.4)$$

As $\mathcal{H}_v\varphi_v' = 0$, zero is in the spectrum of \mathcal{H}_v . We make the following assumption concerning \mathcal{H}_v and $\ddot{d}(v)$:

Assumption 1. (i) *The operator \mathcal{H}_v has a unique negative eigenvalue, a simple eigenvalue 0 and the rest of its spectrum is positive.* (ii) $\ddot{d}(v) > 0$.

Lemma 2. *If Assumption 1 holds and y satisfies $\langle Q'(\varphi_v), y \rangle = \langle \varphi_v', y \rangle = 0$, then $\langle \mathcal{H}_v y, y \rangle > 0$.*

Proof. Differentiating (4.1) with respect to v yields (since c_1 does not depend on v)

$$\mathcal{H}_v \dot{\varphi}_v = -Q'(\varphi_v) \quad (4.5)$$

and from (4.3) we obtain

$$\langle \mathcal{H}_v \dot{\varphi}_v, \dot{\varphi}_v \rangle = -\ddot{d}(v) < 0. \quad (4.6)$$

Putting $y = a_1\chi + p_1$, $p_1 \in P$, where χ is an eigenfunction of \mathcal{H}_v corresponding to the negative eigenvalue $-\lambda_0^2$ and P is the positive subspace of \mathcal{H}_v , we obtain

$$\langle \mathcal{H}_v y, y \rangle = -a_1^2 \lambda_0^2 + \langle \mathcal{H}_v p_1, p_1 \rangle.$$

Write $\dot{\varphi}_v = a_0\chi + b_0\varphi_v' + p_0$, $p_0 \in P$. From (4.6) we have

$$0 > \langle \mathcal{H}_v \dot{\varphi}_v, \dot{\varphi}_v \rangle = -a_0^2 \lambda_0^2 + \langle \mathcal{H}_v p_0, p_0 \rangle$$

and

$$\begin{aligned} 0 &= -\langle Q'(\varphi_v), y \rangle = \langle \mathcal{H}_v \dot{\varphi}_v, y \rangle \\ &= \langle -a_0\lambda_0^2\chi + \mathcal{H}_v p_0, a_1\chi + p_1 \rangle = -a_0a_1\lambda_0^2 + \langle \mathcal{H}_v p_0, p_1 \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \langle \mathcal{H}_v y, y \rangle &= -a_1^2 \lambda_0^2 + \langle \mathcal{H}_v p_1, p_1 \rangle \geq -a_1^2 \lambda_0^2 + \frac{\langle \mathcal{H}_v p_0, p_1 \rangle^2}{\langle \mathcal{H}_v p_0, p_0 \rangle} \\ &= -a_1^2 \lambda_0^2 + \frac{a_0^2 a_1^2 \lambda_0^4}{\lambda_0^2 a_0^2 - \ddot{d}(v)} = a_1^2 K_1. \end{aligned}$$

The rest of the proof is the same as in [29], pages 310–311.

Proposition 5. *If Assumption 1 holds, then for any positive ε there exists $K > 0$ such that for any solution of (1.1) satisfying $u \in \mathcal{U}_\varepsilon$, $Q(u) = Q(\varphi_v)$, one holds*

$$E(u) - E(\varphi_v) + c_1 V(u) - c_1 V(\varphi_v) \geq K \|u(\cdot + \alpha(u)) - \varphi_v\|_1^2. \quad (4.7)$$

Proof. Denote $\psi = u(\cdot + \alpha(u)) - \varphi_v = \mu(\varphi_v - \varphi_v'') + y$ where $\langle \varphi_v - \varphi_v'', y \rangle = 0$. By the translation invariant property of Q , we have

$$\begin{aligned} Q(\varphi_v) = Q(u) &= Q(\varphi_v) + \langle Q'(\varphi_v), \psi \rangle + O(\|\psi\|_1^2) \\ &= Q(\varphi_v) + \langle \varphi_v - \varphi_v'', \mu(\varphi_v - \varphi_v'') + y \rangle + O(\|\psi\|_1^2) \\ &= Q(\varphi_v) + \mu \|\varphi_v - \varphi_v''\|_{L^2}^2 + O(\|\psi\|_1^2). \end{aligned}$$

This implies $\mu = O(\|\psi\|_1^2)$. Since $Q(u) = Q(\varphi_v)$, the identity $M''(\varphi_v) = \mathcal{H}_v$ and the Taylor expansion give

$$E(u) - E(\varphi_v) + c_1 \int_0^T \psi dx = \frac{1}{2} \langle \mathcal{H}_v y, y \rangle + o(\|\psi\|_1^2).$$

From Lemma 1 we have

$$0 = \langle u(\cdot + \alpha(u)), \varphi_v' \rangle = \langle \mu(\varphi_v - \varphi_v'') + y + \varphi_v, \varphi_v' \rangle = \langle y, \varphi_v' \rangle.$$

From the above equality and Lemma 2 we obtain

$$E(u) - E(\varphi_v) + c_1 \int_0^T \psi dx \geq K \|y\|_1^2 + o(\|\psi\|_1^2).$$

This estimate together with

$$\|y\|_1 = \|\psi - \mu(\varphi_v - \varphi_v'')\|_1 \geq \|\psi\|_1 - |\mu| \|\varphi_v - \varphi_v''\|_1$$

yield

$$E(u) - E(\varphi_v) + c_1 \int_0^T \psi dx \geq K \|\psi\|_1^2$$

for $\|\psi\|_1$ sufficiently small. This completes the proof.

Theorem 2. *If Assumption 1 holds, then the travelling-wave solution φ_v is stable.*

Proof. Suppose that φ_v is unstable. Then there exists a sequence of initial data $u_n(0) \in H^1$ and $\eta > 0$ such that

$$\|u_n(0) - \varphi_v\|_1 \rightarrow 0$$

but

$$\sup_{t \in [0, t_0]} \inf_{r \in \mathbb{R}} \|u_n(\cdot, t) - \varphi_v(\cdot - r)\|_1 \geq \eta, \quad (4.8)$$

where $u_n \in C([0, t_0]; H^1)$ is a solution of (1.1) with initial data $u_n(0)$. Let $t_n \in [0, t_0]$ be the first time so that

$$\inf_{r \in \mathbb{R}} \|u_n(\cdot, t_n) - \varphi_v(\cdot - r)\|_1 = \eta.$$

We have

$$\begin{aligned} E(u_n(\cdot, t_n)) &= E(u_n(0)) \rightarrow E(\varphi_v), \\ Q(u_n(\cdot, t_n)) &= Q(u_n(0)) \rightarrow Q(\varphi_v), \\ V(u_n(\cdot, t_n)) &= V(u_n(0)) \rightarrow V(\varphi_v). \end{aligned}$$

Choose a sequence $\psi_n \in H^1$ such that $Q(\psi_n) = Q(\varphi_v)$ and $\|\psi_n - u_n(\cdot, t_n)\|_1 \rightarrow 0$. By continuity of E and V , $E(\psi_n) \rightarrow E(\varphi_v)$ and $V(\psi_n) \rightarrow V(\varphi_v)$. From (4.7) we have

$$E(\psi_n) - E(\varphi_v) + c_1(V(\psi_n) - V(\varphi_v)) \geq K\|\psi_n(\cdot + \alpha(\psi_n)) - \varphi_v\|_1^2.$$

Therefore $\|\psi_n - \varphi_v(\cdot - \alpha(\psi_n))\|_1 \rightarrow 0$, which implies

$$\|u_n - \varphi_v(\cdot - \alpha(\psi_n))\|_1 \rightarrow 0.$$

This however contradicts (4.8). The proof of Theorem 2 is complete.

In order to apply Theorem 2 we have, according to Assumption 1, to determine the sign of the derivative $\ddot{d}(v)$. Using the same technique as in the proof of Proposition 2, we obtain the following expression of $\ddot{d}(v)$ through line integrals. When $a(u)$ and $b(u)$ are polynomials (or even rational functions), these are complete Abelian integrals. A lot of methods have been developed to investigate Abelian integrals, which could be applied here to study the sign of $\ddot{d}(v)$.

Proposition 6. *Assume that the period $T = T(\varphi_0) = \bar{T}(h)$ is not critical where h is given by (3.11). Then*

$$\ddot{d}(v) = \frac{W(h)}{4\bar{T}'(h)}, \quad (4.9)$$

with

$$W(h) = \left(\frac{d}{dh} \oint_{H=h} \frac{dx}{y} \right) \left(\frac{d}{dh} \oint_{H=h} \frac{(x^4 + 2x^2y^2 - \frac{1}{3}y^4)dx}{y} \right) - \left(\frac{d}{dh} \oint_{H=h} \frac{(x^2 + y^2)dx}{y} \right)^2.$$

Proof. We apply again (3.13) and use the Gelfand-Leray form (in both directions) to calculate the needed derivatives. As

$$\dot{d}(v) = \frac{1}{2} \oint_{H=h} \frac{(x^2 + y^2)dx}{y} = \frac{1}{2} \oint_{H=h} y dx + \frac{1}{2} \frac{d}{dh} \oint_{H=h} x^2(v - b(x))y dx$$

one obtains

$$\begin{aligned} \ddot{d}(v) &= \frac{\dot{h}}{2} \frac{d}{dh} \oint_{H=h} \frac{(x^2 + y^2)dx}{y} + \frac{1}{2} \frac{d}{dv} \left(\oint_{H=h} y dx + \frac{d}{dh} \oint_{H=h} x^2(v - b(x))y dx \right) \\ &= \frac{\dot{h}}{2} \frac{d}{dh} \oint_{H=h} \frac{(x^2 + y^2)dx}{y} + \oint_{H=h} \frac{(x^2 - y^2)dx}{4(v - b(x))y} + \frac{d}{dh} \oint_{H=h} \left(\frac{x^2y}{2} + \frac{x^2(x^2 - y^2)}{4y} \right) dx \\ &= \frac{\dot{h}}{2} \oint_{H=h} \frac{(x^2 + y^2)dx}{y} + \frac{1}{4} \frac{d}{dh} \oint_{H=h} \frac{(x^4 + 2x^2y^2 - \frac{1}{3}y^4)dx}{y}. \end{aligned}$$

Replacing the value of \dot{h} from (3.16), we come to the needed formula. \square

5. Examples.

Example 5.1: the BBM equation. Consider the BBM equation

$$u_t + 2\omega u_x + 3uu_x - u_{xxt} = 0, \quad \omega \in \mathbb{R} \quad (\text{BBM})$$

which is a particular case of (1.1) with $a(u) = 2\omega u + \frac{3}{2}u^2$ and $b = 0$. To apply Theorem 2 to BBM, we have to verify Assumption 1. Namely, to establish that the corresponding operator \mathcal{H}_v has the needed spectral properties (i) and to prove the convexity of $d(v)$.

Let us first mention that (3.5) reduces now to $D \equiv (v - 2\omega)^2 + 6c_1 > 0$ and:

$$X_0 = \frac{v - 2\omega + D^{1/2}}{3}, \quad \Delta = \left(\frac{v - 2\omega - D^{1/2}}{3}, \frac{v - 2\omega + 2D^{1/2}}{3} \right) \quad \text{if } v > 0,$$

$$X_0 = \frac{v - 2\omega - D^{1/2}}{3}, \quad \Delta = \left(\frac{v - 2\omega - 2D^{1/2}}{3}, \frac{v - 2\omega + D^{1/2}}{3} \right) \quad \text{if } v < 0.$$

By the definition of c_1, c_2 and $U(s)$ one obtains in the considered case

$$\begin{aligned} U(s) &\equiv \frac{1}{v}(\varphi_0 - s)[s^2 + (\varphi_0 + 2\omega - v)s - (2\varphi_0^2 + (2\omega - v)\varphi_0 + 2v\varphi_2)] \\ &= \frac{(s - \varphi_0)(\varphi_1 - s)(s + \varphi_1 + \varphi_0 + 2\omega - v)}{v}. \end{aligned}$$

We note that the last equality is a consequence of Proposition 1 (ii) which implies that $U(\varphi_1) = U(\varphi_0) = 0$. To obtain an explicit formula for the travelling wave φ_v , we substitute $\sigma = \varphi_0 + (\varphi_1 - \varphi_0)z^2$, $z > 0$ in order to express the integral in (3.6) as an elliptic integral of the first kind in a Legendre form. If $v < 0$, one obtains

$$\int_0^{Z(s)} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} = \alpha(s - \xi)$$

where

$$Z(s) = \sqrt{\frac{\varphi_v(s) - \varphi_0}{\varphi_1 - \varphi_0}}, \quad k^2 = -\frac{\varphi_1 - \varphi_0}{\varphi_1 + 2\varphi_0 + 2\omega - v}, \quad \alpha = \sqrt{\frac{\varphi_1 + 2\varphi_0 + 2\omega - v}{4v}}.$$

Thus we get the expression

$$\varphi_v(s) = \varphi_0 + (\varphi_1 - \varphi_0)sn^2(\alpha(s - \xi); k). \quad (5.1)$$

Similarly, in the case $v > 0$ we obtain (with the same Z)

$$\int_0^{Z(s)} \frac{dz}{\sqrt{(1 - z^2)(k'^2 + k^2 z^2)}} = \alpha(s - \xi)$$

where

$$k^2 = \frac{\varphi_1 - \varphi_0}{\varphi_0 + 2\varphi_1 + 2\omega - v}, \quad k^2 + k'^2 = 1, \quad \alpha = \sqrt{\frac{\varphi_0 + 2\varphi_1 + 2\omega - v}{4v}},$$

and the expression for φ_v

$$\varphi_v(s) = \varphi_0 + (\varphi_1 - \varphi_0)cn^2(\alpha(s - \xi); k). \quad (5.2)$$

To calculate the period of φ_v , we use (3.7) and the same procedure as above. In this way we get in both cases

$$T = 2 \int_{\varphi_0}^{\varphi_1} \frac{d\sigma}{\sqrt{U(\sigma)}} = \frac{2}{\alpha} \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} = \frac{2K(k)}{\alpha}.$$

We return to the operator \mathcal{H}_v defined by (4.4) which now has the form

$$\mathcal{H}_v = -v\partial_x^2 + v - 2\omega - 3\varphi_v \quad (5.3)$$

where φ_v is determined by (5.1) or (5.2). Take $v > 0$ and consider the spectral problem

$$\begin{aligned} \mathcal{H}_v \psi &= \lambda \psi, \\ \psi(0) &= \psi(T), \quad \psi'(0) = \psi'(T). \end{aligned} \quad (5.4)$$

We will denote the operator just defined again by \mathcal{H}_v . It is a self-adjoint operator acting in $H^2([0, T])$. From the Floquet theory applied to (5.4) it follows [39] that its spectrum is purely discrete,

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots \quad (5.5)$$

where λ_0 is always a simple eigenvalue. If $\psi_n(x)$ is the eigenfunction corresponding to λ_n , then

$$\begin{aligned} \psi_0 &\text{ has no zeroes in } [0, T]; \\ \psi_{2n+1}, \psi_{2n+2} &\text{ have each just } 2n + 2 \text{ zeroes in } [0, T]. \end{aligned} \quad (5.6)$$

Proposition 7. *The linear operator \mathcal{H}_v defined by (5.3) – (5.4) has the following spectral properties for $v > 0$:*

- (i) *The first three eigenvalues of \mathcal{H}_v are simple.*
- (ii) *The second eigenvalue of \mathcal{H}_v is $\lambda_1 = 0$.*

Proof. By (3.1), $\mathcal{H}_v \varphi'_v = 0$, hence $\psi = \varphi'_v$ is an eigenfunction corresponding to zero eigenvalue. By Proposition 1 (iii) φ' has just two zeroes in $[0, T)$ and therefore by (5.6) either $0 = \lambda_1 < \lambda_2$ or $\lambda_1 < \lambda_2 = 0$ or $\lambda_1 = \lambda_2 = 0$. We are going to verify that only the first possibility $0 = \lambda_1 < \lambda_2$ can occur. From the definition of k and α one obtains that

$$\varphi_0 + 2\varphi_1 + 2\omega - v = 4v\alpha^2, \quad \varphi_1 - \varphi_0 = 4vk^2\alpha^2.$$

Then using (5.2) we get

$$\begin{aligned}
\mathcal{H}_v &= -v\partial_x^2 + v - 2\omega - 3\varphi_0 - 3(\varphi_1 - \varphi_0)cn^2(\alpha x; k) \\
&= -v\partial_x^2 + v - 2\omega - 3\varphi_1 + 3(\varphi_1 - \varphi_0)sn^2(\alpha x; k) \\
&= -v\partial_x^2 - v\alpha^2[4k^2 + 4 - 12k^2sn^2(\alpha x; k)] \\
&= v\alpha^2[-\partial_y^2 - 4k^2 - 4 + 12k^2sn^2(y; k)] \equiv v\alpha^2\Lambda
\end{aligned}$$

where $y = \alpha x$. The operator Λ is related to Hill's equation with Lamé potential

$$\Lambda w = -\frac{d^2}{dy^2}w + [12k^2sn^2(y; k) - 4k^2 - 4]w = 0$$

and its spectral properties in the interval $[0, 2K(k)]$ are well known [2, 5, 32]. The first three (simple) eigenvalues and corresponding periodic eigenfunctions of Λ are

$$\begin{aligned}
\mu_0 &= k^2 - 2 - 2\sqrt{1 - k^2 + 4k^4} < 0, \\
\psi_0(y) &= dn(y; k)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4})sn^2(y; k)] > 0, \\
\mu_1 &= 0, \\
\psi_1(y) &= dn(y; k)sn(y; k)cn(y; k) = \frac{1}{2}(d/dy)sn^2(y; k), \\
\mu_2 &= k^2 - 2 + 2\sqrt{1 - k^2 + 4k^4} > 0, \\
\psi_2(y) &= dn(y; k)[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4})sn^2(y; k)].
\end{aligned}$$

As the eigenvalues of \mathcal{H}_v and Λ are related by $\lambda_n = v\alpha^2\mu_n$ we conclude that for $v > 0$, the first three eigenvalues of (5.3)-(5.4) are simple and moreover $\lambda_0 < 0$, $\lambda_1 = 0$, $\lambda_2 > 0$. The corresponding eigenfunctions are $\psi_0(\alpha x)$, $\psi_1(\alpha x) = const.\varphi'_v(x)$ and $\psi_2(\alpha x)$. \square

What about the sign of $\ddot{d}(v)$, it is easily seen it is positive when $\omega = 0$ (see the end of this section). For $\omega \neq 0$, the proof is much more long and complicated. It can be done by following the procedure we propose in the next subsection. For this reason, we are not going to investigate the case $\omega \neq 0$ in the present paper.

Example 5.2: the modified BBM equation. Our second example is concerned with the periodic travelling-wave solutions of the equation

$$u_t + 2wu_x + \beta(u^3)_x - u_{xxt} = 0 \quad (\text{mBBM})$$

where $\omega, \beta \in \mathbb{R}$ and $\beta \neq 0$. For this case, $a(u) = 2wu + \beta u^3$ and $b(u) = 0$ in (1.1). For definiteness, we take $\varphi_v = \varphi(x - vt)$ where $v > 0$. The Hamiltonian flow in (3.4) is generated by

$$H(X, Y) = \frac{v}{2}Y^2 + \frac{\beta}{4}X^4 + \left(\omega - \frac{v}{2}\right)X^2 - c_1X.$$

Our plan is to study here the "symmetric" case $c_1 = 0$. The general case could then be considered as a perturbation of the symmetric one (at least for c_1 small). It is well known that there are three cases related to the symmetric Hamiltonian

$$Y^2 + \frac{\beta}{2v}X^4 + \left(\frac{2\omega}{v} - 1\right)X^2, \quad v > 0,$$

see e.g. [31]:

- (i) global center: $\beta > 0, 2\omega > v$;
- (ii) truncated pendulum: $\beta < 0, 2\omega > v$;
- (iii) Duffing oscillator: $\beta > 0, 2\omega < v$,

with three topologically different phase portraits. There is one continuous family of periodic orbits in cases (i) and (ii) and three families (left, right, outer) in (iii) (we advice the reader to draw a picture). This also holds true for all c_1 in case (i) and for $c_1^2 < \frac{4}{27}\beta^{-1}(v - 2\omega)^3$ in (ii), (iii). In the symmetric case $c_1 = 0$ we deal with, the periodic solutions exist if and only if $\varphi_0 \in \Delta^-$ (see Proposition 1) where:

- $\Delta^- = (-\infty, 0)$ in case (i);
- $\Delta^- = (-(\frac{v-2\omega}{\beta})^{1/2}, 0)$ in case (ii);
- $\Delta^- = (-(\frac{2v-4\omega}{\beta})^{1/2}, -(\frac{v-2\omega}{\beta})^{1/2})$ in case (iii), left family;
- $\Delta^- = (0, (\frac{v-2\omega}{\beta})^{1/2})$ in case (iii), right family;
- $\Delta^- = (-\infty, -(\frac{2v-4\omega}{\beta})^{1/2})$ in case (iii), outer family.

The respective functions $U(s)$ take the form:

$$\begin{aligned} U(s) &= \frac{\beta}{2v}(\varphi_0^2 - s^2)(\frac{4\omega-2v}{\beta} + \varphi_0^2 + s^2) && \text{in case (i) and case (iii), outer family,} \\ U(s) &= -\frac{\beta}{2v}(\varphi_0^2 - s^2)(\frac{2v-4\omega}{\beta} - \varphi_0^2 - s^2) && \text{in case (ii),} \\ U(s) &= \frac{\beta}{2v}(s^2 - \varphi_1^2)(\varphi_0^2 - s^2) && \text{in case (iii), left family,} \\ U(s) &= \frac{\beta}{2v}(s^2 - \varphi_0^2)(\varphi_1^2 - s^2) && \text{in case (iii), right family.} \end{aligned}$$

In a similar way as we have done in the (BBM) case, we can use formula (3.6) to calculate explicitly φ_v . By an appropriate change of the variables in (3.6), one can express the solution through standard elliptic integrals (we omit the details). Up to a translation of the argument, we have:

$$\begin{aligned} \varphi_v(s) &= \varphi_0 cn(\alpha s; k), \quad \alpha = \sqrt{\frac{2\omega-v+\beta\varphi_0^2}{v}}, \quad k = \sqrt{\frac{\beta\varphi_0^2}{4\omega-2v+2\beta\varphi_0^2}}, && \text{(i) and (iii) outer,} \\ \varphi_v(s) &= \varphi_0 sn(\alpha s; k), \quad \alpha = \sqrt{\frac{4\omega-2v+\beta\varphi_0^2}{2v}}, \quad k = \sqrt{\frac{\beta\varphi_0^2}{2v-4\omega-\beta\varphi_0^2}}, && \text{(ii),} \\ \varphi_v(s) &= \varphi_0 dn(\alpha s; k), \quad \alpha = \sqrt{\frac{\beta\varphi_0^2}{2v}}, \quad k = \sqrt{\frac{4\omega-2v+2\beta\varphi_0^2}{\beta\varphi_0^2}}, && \text{(iii) left,} \\ \varphi_v(s) &= \varphi_1 dn(\alpha s; k), \quad \alpha = \sqrt{\frac{\beta\varphi_1^2}{2v}}, \quad k = \sqrt{\frac{4\omega-2v+2\beta\varphi_1^2}{\beta\varphi_1^2}}, && \text{(iii) right.} \end{aligned}$$

These formulas of α and k yield the following expressions and range I for the period:

$$T = 4\sqrt{\frac{v}{2\omega-v}}\sqrt{1-2k^2}K(k), \quad k \in (0, \frac{1}{\sqrt{2}}), \quad I = (0, 2\pi\sqrt{\frac{v}{2\omega-v}}), \quad (\text{i}),$$

$$T = 4\sqrt{\frac{v}{2\omega-v}}\sqrt{1+k^2}K(k), \quad k \in (0, 1), \quad I = (2\pi\sqrt{\frac{v}{2\omega-v}}, \infty), \quad (\text{ii}),$$

$$T = 2\sqrt{\frac{v}{v-2\omega}}\sqrt{2-k^2}K(k), \quad k \in (0, 1), \quad I = (2\pi\sqrt{\frac{v}{2v-4\omega}}, \infty), \quad (\text{iii}) \text{ left and right},$$

$$T = 4\sqrt{\frac{v}{v-2\omega}}\sqrt{2k^2-1}K(k), \quad k \in (\frac{1}{\sqrt{2}}, 1), \quad I = (0, \infty), \quad (\text{iii}) \text{ outer}.$$

Indeed, the formulas just derived imply that $T = T(k)$ is strictly decreasing in case (i) and strictly increasing in the other cases. In fact, for (i), (ii) and (iii) outer this follows already from (3.11) and [24]. As far as cases (iii) left and right are concerned, now the result follows from

$$\frac{d}{dk}(\sqrt{2-k^2}K(k)) = \frac{(2-k^2)K' - kK}{\sqrt{2-k^2}} = \frac{K' + E'}{\sqrt{2-k^2}} > 0.$$

On the other hand, in all cases one has $dk/d\varphi_0 \neq 0$. Therefore, given $T \in I$, the condition holds in order to determine φ_0 by the implicit function theorem so that the respective φ would have a period T .

Finally, by using the above formulas one easily obtains that

$$\mathcal{H}_v = -v\partial_x^2 + v - 2\omega - 3\beta\varphi_v^2 = v\alpha^2[-\partial_y^2 + 6k^2\text{sn}^2(y; k) + m]$$

where $y = \alpha x$ and $m = -1 - 4k^2$ in cases (i) and (iii) outer, $m = -4 - k^2$ in cases (iii) left and right, $m = -1 - k^2$ in case (ii), respectively.

Lemma 3. *The first five eigenvalues of the operator Λ defined by the differential expression $\Lambda = -\partial_y^2 + 6k^2\text{sn}^2(y; k)$, with periodic boundary conditions on $[0, 4K(k)]$, are simple. These eigenvalues and their respective eigenfunctions are:*

$$\begin{aligned} \mu_0 &= 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, & \psi_0(y) &= 1 - (1 + k^2 - \sqrt{1-k^2+k^4})\text{sn}^2(y; k), \\ \mu_1 &= 1 + k^2, & \psi_1(y) &= \text{cn}(y; k)\text{dn}(y; k) = \text{sn}'(y; k), \\ \mu_2 &= 1 + 4k^2, & \psi_2(y) &= \text{sn}(y; k)\text{dn}(y; k) = -\text{cn}'(y; k), \\ \mu_3 &= 4 + k^2, & \psi_3(y) &= \text{sn}(y; k)\text{cn}(y; k) = -k^{-2}\text{dn}'(y; k), \\ \mu_4 &= 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, & \psi_4(y) &= 1 - (1 + k^2 + \sqrt{1-k^2+k^4})\text{sn}^2(y; k). \end{aligned}$$

Proof. The equalities $\Lambda\psi_n(y) = \mu_n\psi_n(y)$, $0 \leq n \leq 4$ are established by calculation. By (5.5), (5.6) and the properties of elliptic functions, μ_n are simple and the rest of the spectrum lies in the interval (μ_4, ∞) . \square

Corollary 1. *The first three eigenvalues of the operator Λ , equipped with periodic boundary conditions on $[0, 2K(k)]$, are simple and equal to μ_0, μ_3, μ_4 .*

By Lemma 3 and its corollary, φ'_v is the second eigenfunction of the operator \mathcal{H}_v in the cases: truncated pendulum and Duffing oscillator (left and right). For these cases, it makes sense to investigate the sign of $\ddot{d}(v)$ which we do in the next proposition.

Proposition 8. *Assume that $T \in I = (T_0, \infty)$ and $\varphi_v = \varphi(x - vt)$, $v > 0$, is the periodic travelling-wave solution having a minimal period T . Then:*

- (i) *In the truncated pendulum case one has $\ddot{d}(v) < 0$.*
- (ii) *In the left (right) Duffing oscillator case, if $3v^2 - 8\omega^2 \geq 0$, then $\ddot{d}(v) > 0$. If $3v^2 - 8\omega^2 < 0$ and $2v^2 - 2\omega v - \omega^2 > 0$, then there is $T_{max} \in (T_0, \infty)$ depending only on the ratio ω/v , such that $\ddot{d}(v) > 0$ for $T > T_{max}$ and $\ddot{d}(v) < 0$ for $T \in (T_0, T_{max})$. If $2v^2 - 2\omega v - \omega^2 \leq 0$, then $\ddot{d}(v) < 0$.*

Proof. We are going to use formula (4.9). In the example we deal with, one has

$$y^2 = U(x, h) = \frac{2h}{v} - \frac{2\omega - v}{v}x^2 - \frac{\beta}{2v}x^4, \quad h = \frac{2\omega - v}{2}\varphi_0^2 + \frac{\beta}{4}\varphi_0^4.$$

Given a nonnegative even integer n , denote $I_n(h) = \oint_{H=h} x^n y dx$. It is well known that the linear space of integrals $\{I_n(h), n \text{ even}\}$ forms a polynomial $R[h]$ module with two generators, $I_0(h)$ and $I_2(h)$. Moreover, I_0 and I_2 satisfy a Picard-Fuchs system of dimension two. This implies that the ratio $R(h) = I'_2(h)/I'_0(h)$ satisfies a Riccati equation. We shall use these facts to express $\ddot{d}(v)$ as a quadratic form with respect to I'_0, I'_2 with polynomial coefficients in h and use the properties of the Riccati equation to determine the sign of $\ddot{d}(v)$. The procedure might seem too long, but it is universal (at least in the case when $a(u)$ and $b(u)$ are polynomials) and therefore applicable to many other cases.

Let us first express $\ddot{d}(v)$ through the integrals I_n . Below, we will denote for short the derivatives with respect to h by I'_n, I''_n etc. Using the first equality in (3.13) with $b = 0$, we obtain

$$I'_n(h) = \oint_{H=h} \frac{x^n dx}{vy}.$$

On the other hand,

$$\oint_{H=h} y^3 dx = \oint_{H=h} U(x, h) y dx = \frac{2h}{v} I_0 - \frac{2\omega - v}{v} I_2 - \frac{\beta}{2v} I_4.$$

By using these expressions, we obtain that

$$\ddot{d}(v) = \frac{v}{4I''_0} \left[I''_0 \left(I'_4 + \frac{\beta}{6v^2} I_4 + \frac{2\omega + 5v}{3v^2} I_2 - \frac{2h}{3v^2} I_0 \right)' - \left(\frac{1}{v} I'_0 + I'_2 \right)^2 \right]. \quad (5.7)$$

For reader's convenience, below we proceed to derive the relations between integrals I_n and the Picard-Fuchs system satisfied by I_0 and I_2 .

Lemma 4. (i) *The following relations hold:*

$$(n+6)\beta I_{n+3} + (2n+6)(2\omega - v)I_{n+1} = 4nhI_{n-1}, \quad n = 1, 3, 5, \dots \quad (5.8)$$

(ii) The integrals I_0 and I_2 satisfy the system

$$\begin{aligned} 4hI'_0 - (2\omega - v)I'_2 &= 3I_0, \\ -\frac{4(2\omega - v)}{3\beta}hI'_0 + \left(4h + \frac{4(2\omega - v)^2}{3\beta}\right)I'_2 &= 5I_2. \end{aligned} \quad (5.9)$$

Proof. (i). Integrating by parts, we obtain the identity

$$\oint_{H=h} [x^n U'(x) + \frac{2}{3}nx^{n-1}U(x)]ydx = 0. \quad (5.10)$$

Indeed,

$$\begin{aligned} \oint_{H=h} x^n U'(x)ydx &= \oint_{H=h} x^n y dy^2 = \frac{2}{3} \oint_{H=h} x^n dy^3 = -\frac{2}{3}n \oint_{H=h} x^{n-1}y^3 dx \\ &= -\frac{2}{3}n \oint_{H=h} x^{n-1}U(x)ydx. \end{aligned}$$

Identity (5.10) is equivalent to (5.8).

(ii) Similarly, one has

$$\begin{aligned} \oint_{H=h} \frac{x^n U'(x)dx}{y} &= \oint_{H=h} \frac{x^n dy^2}{y} = \oint_{H=h} 2x^n dy = -2n \oint_{H=h} x^{n-1}ydx, \\ \oint_{H=h} \frac{x^n U(x)dx}{y} &= \oint_{H=h} x^n ydx. \end{aligned} \quad (5.11)$$

Rewriting these identities by means of I_n , we come to the formulas

$$\begin{aligned} \beta I'_{n+3} + (2\omega - v)I'_{n+1} &= nI_{n-1}, \\ -\beta I'_{n+4} - 2(2\omega - v)I'_{n+2} + 4hI'_n &= 2I_n. \end{aligned} \quad (5.12)$$

The last two relations imply

$$4hI'_n - (2\omega - v)I'_{n+2} = (n + 3)I_n. \quad (5.13)$$

Using (5.13) with $n = 0, 2$, we obtain the system

$$\begin{aligned} 4hI'_0 - (2\omega - v)I'_2 &= 3I_0, \\ 4hI'_2 - (2\omega - v)I'_4 &= 5I_2. \end{aligned}$$

We remove I'_4 from the last equation by using the first equation in (5.12). As a result one obtains (5.9). \square

After proving Lemma 4, we return to our main goal. Using the identities

$$I_4 = \frac{4h}{7\beta}I_0 - \frac{8(2\omega - v)}{7\beta}I_2, \quad I'_4 = \frac{1}{\beta}I_0 - \frac{2\omega - v}{\beta}I'_2.$$

and the first equation in (5.9), we remove I_4 , I'_4 and I_0 from (5.7). The result is

$$\ddot{d}(v) = \frac{v}{4I_0''} \left[I_0'' \left(\frac{3v^2 - 4\beta h}{3\beta v^2} I_0' + \frac{2\omega + 5v}{3v^2} I_2' - \frac{2\omega - v}{\beta} I_2'' \right) - \left(\frac{1}{v} I_0' + I_2'' \right)^2 \right]. \quad (5.14)$$

Now, we differentiate (5.9) and determine the second derivatives from the obtained system:

$$\begin{aligned} \mathcal{D}(h)I_0'' &= -4hI_0' + (2\omega - v)I_2', \\ \mathcal{D}(h)I_2'' &= \frac{4(2\omega - v)}{\beta} hI_0' + 4hI_2', \\ \mathcal{D}(h) &= 16h^2 + \frac{4(2\omega - v)^2}{\beta} h. \end{aligned} \quad (5.15)$$

Replacing in (5.14) and performing some direct calculations to simplify the result, we derive the formula

$$\ddot{d}(v) = \frac{1}{9vI_0} \left[\left(8h^2 + \frac{12\omega^2}{\beta} h \right) I_0'^2 + (4\omega + 10v)hI_0'I_2' + (2v^2 - 2\omega v - \omega^2)I_2'^2 \right].$$

By (5.15), the ratio $R(h) = I_2'(h)/I_0'(h)$ satisfies the Riccati equation

$$\mathcal{D}(h)R' = \frac{4(2\omega - v)}{\beta} h + 8hR - (2\omega - v)R^2. \quad (5.16)$$

In order to remove the parameters from (5.16), we take new variables \bar{h} , \bar{R} through

$$h = -\frac{(2\omega - v)^2}{8\beta}(\bar{h} + 1), \quad R(h) = -\frac{2\omega - v}{2\beta}(\bar{R}(\bar{h}) + 1).$$

Thus we obtain the final formulas

$$\ddot{d}(v) = \frac{(2\omega - v)^2 I_0'^2(h)}{72v\beta^2 I_0(h)} w(\bar{h}, \bar{R})$$

where (below we will omit thoroughly the bars)

$$w(h, R) = (4v^2 - 4\omega v - 2\omega^2)R^2 + (4\omega^2 + 8\omega v - 5v^2)hR + (2\omega - v)^2 h^2 + 3v^2(R - h) - 6\omega^2 \quad (5.17)$$

and $R = R(h)$ satisfies the equation (and related system with respect to a dummy variable)

$$\begin{aligned} 4(1 - h^2)R' &= 1 - 2hR + R^2, & \dot{h} &= 4(1 - h^2), \\ & & \dot{R} &= 1 - 2hR + R^2. \end{aligned} \quad (5.18)$$

Since $v > 0$ and $I_0 > 0$ (this is the area integral), the sign of $\ddot{d}(v)$ is determined by w . The conic curve $w(h, R) = 0$ divides the \mathbb{R}^2 -plane into two parts W_+ and W_- according to the sign of $w(h, R)$. We have to identify the curve Γ in the phase portrait of (5.18) which corresponds to I_2'/I_0' and determine its location with respect to W_+

and W_- . In the case when $\ddot{d}(v)$ changes sign, the main problem will be to prove that the conic curve intersects Γ at most once.

It is easy to see that system (5.18) has two critical points $(1, 1)$ and $(-1, -1)$ which are saddle-nodes. They are connected by two separatrix trajectories, upper Γ_u and lower Γ_l , please draw a picture. Obviously, the phase portrait of (5.18) is symmetric with respect to the origin.

Lemma 5. *In (5.18), the trajectory corresponding to the truncated pendulum case is Γ_l . The trajectory corresponding to the left and right Duffing oscillator cases is Γ_u .*

Proof. If $\varphi_0 \in \Delta_-$, then $\varphi_0^2 \in (0, \frac{v-2\omega}{\beta})$ in the truncated pendulum and right Duffing oscillator cases and $\varphi_0^2 \in (\frac{v-2\omega}{\beta}, \frac{2v-4\omega}{\beta})$ in the left case. Correspondingly, one obtains

$$h \in \left(0, -\frac{(2\omega - v)^2}{4\beta}\right) = (h_c, h_s) \text{ in the truncated pendulum case,}$$

$$h \in \left(-\frac{(2\omega - v)^2}{4\beta}, 0\right) = (h_c, h_s) \text{ in the left and right Duffing oscillator cases,}$$

where h_c is the Hamiltonian level corresponding to a center and h_s – to a saddle. Moreover, $R(h) = I_2'(h)/I_0'(h)$ is analytic in a neighborhood of $h = h_c$ and, by the mean value theorem,

$$\lim_{h \rightarrow h_c} R(h) = \lim_{h \rightarrow h_c} \frac{I_2(h)}{I_0(h)} = X_0^2.$$

We recall that X_0 is the abscissa of the center and $X_0 = 0$ in the truncated pendulum case, $X_0^2 = \frac{v-2\omega}{\beta}$ in the left and right Duffing oscillator case. On its turn, by Picard-Lefschetz theory ([6], Chapter 3), near $h = h_s$ the integral $I_n(h)$ has the expansion

$$I_n(h) = I_n(h_s) + \alpha_n(h - h_s) \log |h - h_s| + \beta_n(h - h_s) + \gamma_n(h - h_s)^2 \log |h - h_s| + \dots,$$

where $\alpha_0 \neq 0$. Therefore $R(h) = I_2'(h)/I_0'(h)$ is bounded near $h = h_s$. In terms of the new coordinates (\bar{h}, \bar{R}) , all this means that the phase trajectory of (5.18) corresponding to the truncated pendulum case is $(h, R(h))$ where $-1 < h < 1$, $R(h) \rightarrow \pm 1$ as $h \rightarrow \pm 1$, and $R(h)$ is analytic near $h = -1$. The unique phase curve with these properties is Γ_l . Similarly, the phase trajectory of (5.18) corresponding to the left and right Duffing oscillator cases has the same ends but is analytic near $h = 1$, hence it is Γ_u . \square

Below, we list some properties of the separatrices Γ_l , Γ_u and the conic curve $w = 0$. Apart of the separatrices, the conic curve depends on one parameter ω/v and undergoes several bifurcations when ω/v varies.

Lemma 6. *The separatrices Γ_u and Γ_l have the following properties:*

- (i) Γ_u is increasing and concave, Γ_l is increasing and convex.
- (ii) The tangent to Γ_u at $(-1, -1)$ is $h = -1$, the tangent to Γ_l at $(1, 1)$ is $h = 1$.
- (iii) The tangent to Γ_u at $(1, 1)$ and the tangent to Γ_l at $(-1, -1)$ have a slope $\frac{1}{4}$.

Proof. The proof easily follows from (5.18). Claim (iii) is a consequence of the analyticity at the corresponding point. To establish (ii), we use the expansions containing logarithmic terms above. They imply that the asymptotic expansion near $(-1, -1)$ of the non-analytic trajectories has the form $-1 + \mu/\log(h+1) + \dots$. We determine μ from (5.18) and obtain that on Γ_u one holds

$$R(h) = -1 - \frac{8}{\log(h+1)} + O\left(\log^{-2}(h+1)\right). \quad (5.19)$$

A similar formula holds for Γ_l , hence (ii) is proved. Finally, in the strip $|h| < 1$ one has $1 - 2hR + R^2 > 0$, therefore $R' > 0$ and all trajectories increase. Differentiating with respect to h the Riccati equation in (5.18), we determine the curve of inflection points $R''(h) = 0$ in the phase portrait, which is $(R-h)(R^2 + 2hR - 3) = 0$. As $R' = \frac{1}{4}$ on the line $R = h$, by (ii) and (iii) Γ_u and Γ_l cannot intersect this line. Hence, Γ_u lies in the domain corresponding to the concave trajectories, and Γ_l – to the convex ones. \square

The proof of the statements listed below is obvious.

Lemma 7. *The conic curve $w(h, R) = 0$ has the following properties:*

- (i) *It goes through the critical points $(1, 1)$ and $(-1, -1)$ of (5.18) and $w(h, h) = 6\omega^2(h^2 - 1)$.*
- (ii) *For $\omega = 0$ it degenerates into $v^2(R-h)(4R-h+3) = 0$. For $\omega^2 + 2\omega v - 2v^2 = 0$ it degenerates into $v^2 h[(R-1) + (7 \mp 4\sqrt{3})(h-1)] = 0$.*
- (iii) *If $(h, R(h))$ are the local coordinates near $(1, 1)$, then $R'(1) = 1 - \frac{2\omega^2}{v^2}$, $R''(1) = \frac{8\omega^4(\omega^2 + 2\omega v - 2v^2)}{3v^6}$.*
- (iv) *If $(h(R), R)$ are the local coordinates near $(-1, -1)$ and $\omega \neq 0$, then $h'(-1) = 0$, $h''(-1) = \frac{2v^2 - 2\omega v - \omega^2}{3\omega^2}$.*

After the preparation done above, we proceed to prove statements (i) and (ii) of Proposition 8. To prove (i), let us denote by Ω the open triangle in the (h, R) -plane having vertices at $(1, 1)$, $(-1, -1)$ and $(1, -\frac{1}{2})$. By Lemma 6, $\Gamma_l \subset \Omega$. On its turn, $\Omega \subset W_-$. This holds because by Lemma 7 one has $w(h, h) < 0$ for $|h| < 1$, $w(1, 1) = w(-1, -1) = 0$, $R'(1) < 1$, $h'(-1) = 0$, and because $w(1, -\frac{1}{2}) = -\frac{9}{2}\omega(\omega + 2v) < 0$ (recall that $2\omega > v > 0$ in the truncated pendulum case). Proposition 8 (i) is established.

The proof of (ii) is more complicated. In this case, $2\omega < v$. Assume first that $\omega^2 + 2\omega v - 2v^2 > 0$. Then, by Lemma 7 (iii), (iv) and convexity, the conic curve is a hyperbola. One of its branches is contained in the half-plane $h \leq -1$, the other branch lies above its tangent line at $(1, 1)$, namely $R - 1 = R'(1)(h - 1)$. As $R'(1) < 0$ in the considered case, by Lemma 6 (i) and Lemma 7 (i) we conclude that $\Gamma_u \subset W_-$. The same proof goes in the degenerate case $\omega^2 + 2\omega v - 2v^2 = 0$.

The case $\omega^2 + 2\omega v - 2v^2 < 0$ is more delicate. If $\omega = 0$, then Lemma 6 (i) and Lemma 7 (ii) imply that $\Gamma_u \subset W_+$. Below we will take $\omega \neq 0$. Consider the branch $R = r(h)$ of the conic curve going through both $(1, 1)$ and $(-1, -1)$. Writing

$w(h, R) = \delta_0 R^2 + \delta_1 R + \delta_2$, $\delta_0 > 0$, one obtains

$$r(h) = \frac{-\delta_1 + \sqrt{D}}{2\delta_0}, \quad D = \delta_1^2 - 4\delta_0\delta_2 > 0, \quad |h| < 1. \quad (5.20)$$

Below we will denote this branch by C . By (5.19) and Lemma 7 (iv) we have near $h = -1$

$$R + 1 \sim -\frac{8}{\log(h + 1)} \quad \text{on } \Gamma_u, \quad R + 1 \sim \sqrt{\frac{6\omega^2(h + 1)}{2v^2 - 2\omega v - \omega^2}} \quad \text{on } C.$$

This yields that for h close to -1 , Γ_u is placed above C . Similarly, by Lemma 6 (iii) and Lemma 7 (iii) one obtains that near $h = 1$, Γ_u lies above C if $3v^2 - 8\omega^2 \geq 0$ and below C otherwise. Therefore, to finish the proof of Proposition 7, we have to establish the following: 1) Γ_u is entirely placed above C if $3v^2 - 8\omega^2 \geq 0$, and 2) Γ_u intersects the conic just once if $3v^2 - 8\omega^2 < 0$. In the first case, we would have $\Gamma_u \subset W_+$, while in the second one, the part of Γ_u near $h = 1$ would be in W_- and the remaining part – in W_+ . Unfortunately, both Γ_u and C are concave and it is not so easy to determine the number of their intersections.

Below, we proceed to determine the number of contact points that C has with the vector field (5.18). Because of the type of critical points, in our case the number of intersections is less than or equal to the number of contact points. As well known, the equation of the contact points is given by

$$\frac{d}{ds}(R - r(h))|_{R=r(h)} = [\dot{R} - \dot{h}r'(h)]|_{R=r(h)} = 1 - 2hr(h) + r^2(h) - 4(1 - h^2)r'(h) = 0. \quad (5.21)$$

We replace $r' = -(\delta_1' r + \delta_2')/(2\delta_0 r + \delta_1)$ in (5.21) and use once again the quadratic equation satisfied by r to obtain

$$\left[\delta_1 + 4\delta_2 h + \frac{\delta_1 \delta_2}{\delta_0} + 4\delta_2'(1 - h^2) \right] + r \left[2\delta_0 + 2\delta_1 h - 2\delta_2 + \frac{\delta_1^2}{\delta_0} + 4\delta_1'(1 - h^2) \right] = 0.$$

Next, replacing r from (5.20) and performing the needed calculations we get

$$[12\delta_0(2\omega v - v^2)(1 - h^2) + D]\sqrt{D} - [3v^2(1 + h)D + 2\delta_0(1 - h^2)D'] = 0.$$

By (5.20) and (5.17) one has

$$D = (1 + h)^2(9v^4 + 24\omega^2\delta_0\zeta), \quad D' = (1 + h)(18v^4 + 24\omega^2\delta_0(\zeta - 1)), \quad \zeta = \frac{1 - h}{1 + h} > 0.$$

We replace these values in the equation above and divide the result by $(1 + h)^3$. One obtains

$$\begin{aligned} & [9v^4 + 12(2\omega^2 + 2\omega v - v^2)\delta_0\zeta]\sqrt{9v^4 + 24\omega^2\delta_0\zeta} \\ & - [27v^6 + 12(3v^4 + 6\omega^2v^2 - 4\omega^2\delta_0)\delta_0\zeta + 48\omega^2\delta_0^2\zeta^2] = 0. \end{aligned}$$

At the end, we introduce a new variable z by

$$\sqrt{9v^4 + 24\omega^2\delta_0\zeta} = 3(v^2 + z), \quad z \in (0, \infty)$$

to obtain the equation of contact points in a final form $P(z) = 0$ where

$$\begin{aligned} P(z) &= 3z^3 + 6(3v^2 - 2\omega v - 2\omega^2)z^2 \\ &\quad + 4(v + \omega)(9v^3 - 18\omega v^2 + 4\omega^2 v + 4\omega^3)z + 2v^2\delta_0(3v^2 - 8\omega^2). \end{aligned}$$

Taking $P(z) = 3z^3 + A_2z^2 + A_1z + A_0$, it is easy to verify that:

$$A_2 > 0, A_1 > 0, A_0 > 0 \text{ for } 3v^2 - 8\omega^2 > 0;$$

$$A_2 > 0, A_1 > 0, A_0 < 0 \text{ for } 3v^2 - 8\omega^2 < 0, v + \omega > 0;$$

$$A_1 < 0, A_0 < 0 \text{ for } v + \omega < 0;$$

(recall that $v - 2\omega > 0$ and $\delta_0 = 4v^2 - 4\omega v - 2\omega^2 > 0$). By Descartes chain rule, the equation $P(z) = 0$ has no positive root if $3v^2 - 8\omega^2 \geq 0$ and has exactly one positive root if $3v^2 - 8\omega^2 < 0$ and $2v^2 - 2\omega v - \omega^2 > 0$. All changes of the variables we used throughout the proof were one-to-one, hence there is no contact point in the first case and there is just one contact point in the second case. As a result, in the first case C does not intersect Γ_u and therefore $\Gamma_u \subset W_+$. In the second case, C intersects Γ_u at a unique point corresponding to some $\bar{h}_0 \in (-1, 1)$ so that the part of Γ_u related to $(-1, \bar{h}_0)$ is in W_+ and the remaining part is in W_- . Let φ_0 be the value corresponding to h_0 according to (3.11) and let T_{max} be the period of the orbit going through $(\varphi_0, 0)$ in the (φ, φ') -plane. Then the orbits from the period annulus having a period $T > T_{max}$ will be stable and the remaining ones unstable. Proposition 8 is completely proved. \square .

Let us recall again that in the left(right) Duffing oscillator case the period T belongs to the interval $I = (T_0, \infty)$ where $T_0 = 2\pi\sqrt{\frac{v}{2v-4\omega}}$. Lemma 3 and Proposition 8 imply:

Corollary 2. *Both conditions of Assumption 1 are satisfied in the left(right) Duffing oscillator case, provided that:*

$$(i) \ 3v^2 - 8\omega^2 \geq 0;$$

$$(ii) \ 3v^2 - 8\omega^2 < 0, \ 2v^2 - 2\omega v - \omega^2 > 0 \text{ and the period } T \text{ is sufficiently large.}$$

Proof of Theorem I. Theorem I is a direct consequence of Corollary 2 and Theorem 2, taking into account that φ does not oscillate around zero only in the left and right Duffing oscillator cases. \square

Example 5.3: Coherent single power nonlinearities. Let $a(u) = \beta u^{k+1}$, $b(u) = \gamma \beta u^k$, $k \in \mathbb{N}$, where $|\gamma| < 1$ and $v/\beta > 0$. Or more generally, one can take $a(u) = 2\omega u + \beta u^{k+1}$, $b(u) = 2\omega + \gamma \beta u^k$, where $r = (v - 2\omega)/\beta > 0$. For simplicity, assume that $c_1 = 0$. Then (3.5) is satisfied with $X_0 = r^{1/k}$ and Proposition 1 works. Next, equation (3.3) implies that φ depends on r alone. Moreover, one has

$$\varphi = r^{1/k}\bar{\varphi}, \quad \varphi_0 = r^{1/k}\bar{\varphi}_0, \quad \varphi_1 = r^{1/k}\bar{\varphi}_1, \quad c_2 = r^{\frac{k+2}{k}}\bar{c}_2, \quad U(\varphi) = r^{\frac{2}{k}}\bar{U}(\bar{\varphi}),$$

where $\bar{\varphi}$, $\bar{\varphi}_0$, $\bar{\varphi}_1$, \bar{c}_2 and \bar{U} do not depend on v , ω and β . Therefore we obtain

$$\ddot{d}(v) = \frac{1}{2} \frac{d}{dv} \int_0^T (\varphi^2 + \varphi'^2) dx = \left(\frac{1}{2} \frac{d}{dv} r^{2/k} \right) \int_0^T (\bar{\varphi}^2 + \bar{\varphi}'^2) dx$$

$$= \frac{r^{2/k}}{k(v-2\omega)} \int_0^T (\bar{\varphi}^2 + \bar{\varphi}'^2) dx = \frac{1}{k(v-2\omega)} \int_0^T (\varphi^2 + \varphi'^2) dx. \quad (5.22)$$

Therefore $\ddot{d}(v)$ takes the sign of β (and $v-2\omega$).

6. The perturbed gBBM equation.

Let us denote by $\mathcal{C}(X, Y)$ (respectively, $\mathcal{B}(X, Y)$) the set of all closed (respectively, bounded) operators $S : X \rightarrow Y$. When $X = Y$, we shall write simply $\mathcal{C}(X)$ and $\mathcal{B}(X)$. If $S \in \mathcal{C}(X, Y)$, denote by $\mathbf{G}(S) \subset X \times Y$ its graph. Below we choose $X = Y = L^2[0, T]$. One can define a metric $\hat{\delta}$ on $\mathcal{C}(L^2[0, T])$ as follows: for $R, S \in \mathcal{C}(L^2[0, T])$, set

$$\hat{\delta}(R, S) = \|P_R - P_S\|_{\mathcal{B}(L^2 \times L^2)}$$

where P_R and P_S are the orthogonal projections on the graphs of $\mathbf{G}(R)$ and $\mathbf{G}(S)$, respectively.

The following results are well known (see Kato [34], Chapter 4, Theorems 2.14 and 2.17):

Theorem A. *Take $X = Y = L^2[0, T]$ and assume that $R, S \in \mathcal{C}(X, Y)$, $A \in \mathcal{B}(X, Y)$. Then*

$$\hat{\delta}(R + A, S + A) \leq 2(1 + \|A\|^2)\hat{\delta}(R, S).$$

Theorem B. *Assume that $S \in \mathcal{C}(X, Y)$ and B is a S -bounded operator satisfying $\|Bf\| \leq a\|f\| + b\|Sf\|$ with $b < 1$. Then*

$$R = S + B \in \mathcal{C}(X, Y) \text{ and } \hat{\delta}(R, S) \leq \frac{\sqrt{a^2 + b^2}}{1 - b}.$$

Let us introduce a small perturbation in (gBBM) by taking $a(u) = 2\omega u + \beta u^k$, $k = 2, 3$ and $b(u) = \gamma g(u)$ in the general equation (1.1) where γ is a small real parameter. We will denote the travelling-wave solution corresponding to (1.1) again by φ_v and the one corresponding to (gBBM) (that is, when $\gamma = 0$) by φ_v^0 . If T is the period φ_v^0 , then by Proposition 3 for sufficiently small γ there is a smooth function $\varphi_0(\gamma)$ such that the wave solution φ_v of the perturbed equation which goes through the point $(\varphi_0(\gamma), 0)$ in the (φ, φ') -plane will have the same period T . For the related operators \mathcal{H}_v we have respectively

$$\begin{aligned} \mathcal{H}_v &= -v\partial_x^2 + v - a'(\varphi_v) + \gamma[g(\varphi_v)\partial_x^2 + g'(\varphi_v)\varphi_v'\partial_x + \frac{1}{2}g''(\varphi_v)\varphi_v'^2 + g'(\varphi_v)\varphi_v''], \\ \mathcal{H}_v^0 &= -v\partial_x^2 + v - a'(\varphi_v^0). \end{aligned}$$

Theorem 3. *If Assumption 1(i) holds for the operator \mathcal{H}_v^0 , then for γ sufficiently small it also holds for the operator \mathcal{H}_v .*

Proof. In order to apply Theorem A, we define the operator A to be a multiplication by the function $-a'(\varphi_v^0)$ and take $S = -v\partial_x^2 + v$. Then denoting $B_\gamma = \mathcal{H}_v - \mathcal{H}_v^0$, we obtain from Theorem A the estimate

$$\hat{\delta}(\mathcal{H}_v, \mathcal{H}_v^0) = \hat{\delta}(S + B_\gamma + A, S + A) \leq 2(1 + \|A\|^2)\hat{\delta}(S + B_\gamma, S). \quad (6.1)$$

As

$$B_\gamma = a'(\varphi_v^0) - a'(\varphi_v) + \gamma[-v^{-1}g(\varphi_v)S + g'(\varphi_v)\varphi'_v\partial_x + G]$$

where

$$G = g(\varphi_v) + \frac{1}{2}g''(\varphi_v)\varphi_v'^2 + g'(\varphi_v)\varphi_v'',$$

for $f \in D(S) = H_{per}^2$, we further have

$$\begin{aligned} \|B_\gamma f\| &\leq \max_{[0,T]} |a'(\varphi_v^0) - a'(\varphi_v)| \cdot \|f\| \\ &\quad + |\gamma| [\max_{[0,T]} |g(\varphi_v)/v| \cdot \|Sf\| + \max_{[0,T]} |g'(\varphi_v)\varphi'_v| \cdot \|\partial_x f\| + \max_{[0,T]} |G| \cdot \|f\|]. \end{aligned}$$

From Plancherel identity, we have

$$\begin{aligned} \|Sf\|^2 &= \int_0^T Sf \cdot \overline{Sf} dx = T \sum_{k=-\infty}^{+\infty} |\widehat{Sf}(k)|^2 \\ &= v^2 \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^2 |\widehat{f}(k)|^2 \geq K(v) \sum_{k=-\infty}^{+\infty} |k|^2 |\widehat{f}(k)|^2 = K(v) \|\partial_x f\|^2 \end{aligned}$$

where $K(v)$ is a constant depending only on v . By using Propositions 3 and 4, we obtain that $\max_{[0,T]} |a'(\varphi_v^0) - a'(\varphi_v)| = O(\gamma)$. Taking into account these estimates, we come to the conclusion that the operator $B = B_\gamma$ is S -bounded, with constants a, b tending to zero as $\gamma \rightarrow 0$. Applying Theorem B and inequality (6.1), we obtain

$$\hat{\delta}(\mathcal{H}_v, \mathcal{H}_v^0) \leq 2(1 + \|A\|^2) \frac{\sqrt{a^2 + b^2}}{1 - b} = O(\gamma). \quad (6.2)$$

Therefore by [34] (Theorem 3.16 in Chapter 4), the operators \mathcal{H}_v and \mathcal{H}_v^0 have the same spectral properties. More precisely (see also [1], p. 363), let U is the open disc bounded by Γ and $\text{spec}(\mathcal{H}_v^0) \cap \overline{U} = \{\alpha_0, 0\}$, where α_0 is the negative eigenvalue of \mathcal{H}_v^0 . Choose circular contours Γ_1 and Γ_2 contained in U , such that if U_1 and U_2 are the open discs bounded by Γ_1 and Γ_2 respectively, then $\text{spec}(\mathcal{H}_v^0) \cap \overline{U}_1 = \{\alpha_0\}$ and $\text{spec}(\mathcal{H}_v^0) \cap \overline{U}_2 = \{0\}$. From (6.2) for sufficiently small γ we have that the $\text{spec}(\mathcal{H}_v) \cap \overline{U}_1$ and $\text{spec}(\mathcal{H}_v) \cap \overline{U}_2$ consist of a single, simple eigenvalue. Since 0 is an eigenvalue of \mathcal{H}_v we must have $\text{spec}(\mathcal{H}_v) \cap \overline{U}_2 = \{0\}$, which shows that 0 is a simple eigenvalue of \mathcal{H}_v . Similarly we obtain that $\text{spec}(\mathcal{H}_v) \cap \overline{U}$ consists of a finite set of eigenvalues of total multiplicity 2. This completes the proof of the theorem.

Proof of Theorems II and III. Take $\omega = 0$. Inequality (5.22) implies that $\ddot{d}(v)$ is positive. When $\gamma = 0$, applying respectively Proposition 7 and Corollary 1, we conclude that \mathcal{H}_v^0 has the needed spectral properties in both cases. By Theorem 3 the same remains true for small $|\gamma|$ as well. Hence, Theorem 2 applies to both cases. \square

7. Small-amplitude travelling-wave solutions.

Since the expressions in (3.10) and (4.9) are too complex, it is difficult to use them in general. Below we are going to consider the simpler case when the periodic waves we study are of small amplitude. That is, $\varphi_1 - \varphi_0$ is close to zero and therefore the periodic trajectory is entirely contained in a small neighborhood of some center $(X_0, 0) \in \mathbb{R}^2$ given by (3.5). Below we establish that an important role in the stability of the small-amplitude travelling-wave solutions is played by the first isochronous (or period) constant. Let us recall that the period function has an expansion

$$T(r) = T_0 + T_2 r^2 + T_4 r^4 + T_6 r^6 + T_8 r^8 + \dots$$

with respect to r , the distance between the center at $(X_0, 0)$ and the intersection point of the orbit with the x -axis $(\varphi_0, 0)$. Then T_{2k} , $k = 1, 2, \dots$ is the k th isochronous constant. When all period constants are zero, the center is isochronous and all orbits surrounding it have the same period T_0 . Since $T'(\varphi_0) = -dT/dr$, the above expansion yields immediately:

Proposition 9. *The period of a small-amplitude travelling-wave solution around a non-isochronous center is not critical.*

Let us mention, however, that the calculation of the isochronous constants is a difficult task and the problem of isochronicity is completely solved for only few particular cases. Denote for short

$$p = -\frac{a''(X_0)}{3(a'(X_0) - v)}, \quad q = -\frac{a'''(X_0)}{3(a'(X_0) - v)}, \quad m = \frac{b'(X_0)}{v - b(X_0)}, \quad n = \frac{b''(X_0)}{v - b(X_0)}. \quad (7.1)$$

As we shall see below, the first isochronous constant in our case is expressed by

$$T_2 = T_0 \vartheta, \quad T_0 = 2\pi \sqrt{\frac{v - b(X_0)}{a'(X_0) - v}}, \quad \vartheta = \frac{1}{16}(15p^2 + 3q - m^2 - 6pm - 2n).$$

Take a small positive ε and let $\varphi_0 = X_0 - \varepsilon$. Then by (3.9) we obtain

$$\varphi_1 = X_0 + \varepsilon + p\varepsilon^2 + p^2\varepsilon^3 + O(\varepsilon^4). \quad (7.2)$$

The expression (7.2) as well as the formulas that follow are obtained by long and boring asymptotic calculations which we will omit here. Thus, by using (3.8) we come to the expression

$$U(X) = \frac{\varepsilon^2(a'(X_0) - v)(1 - s^2)[1 + p(1 - s)\varepsilon - \frac{1}{4}(p^2(1 + 6s) + q(1 + s^2))\varepsilon^2 + \dots]}{(v - b(X_0))[1 - ms\varepsilon - \frac{1}{2}(mp(1 + s) + ns^2)\varepsilon^2 + \dots]}$$

Then, by (3.7) and (3.8) we obtain

$$T = T_0[1 + \vartheta\varepsilon^2 + O(\varepsilon^4)]. \quad (7.3)$$

In order to calculate the sign of the derivative $\ddot{d}(v)$, we prefer to apply a more direct approach, deriving first an alternative formula instead of (4.9). By using (4.3), we obtain

$$\ddot{d}(v) = \frac{d}{dv} \left(\int_0^{T/2} (\varphi^2 + \varphi'^2) dx \right) = \frac{d}{dv} \left(\int_{\varphi_0}^{\varphi_1} [s^2 + U(s)] \frac{ds}{\sqrt{U(s)}} \right).$$

In the calculation below, we denote by $\dot{\varphi}_0, \dot{\varphi}_1$ etc. the derivatives with respect to the parameter v . As $U(s)$ vanishes at φ_0 and φ_1 , then clearly

$$\frac{d}{dv} \left(\int_{\varphi_0}^{\varphi_1} \sqrt{U(s)} ds \right) = \int_{\varphi_0}^{\varphi_1} \frac{(\dot{U}(s) + \dot{\varphi}_0 U_{\varphi_0}(s)) ds}{2\sqrt{U(s)}}.$$

To perform differentiation in the other part of the integral, we first use a change of the variable like (3.8) and return to the initial variable after the calculations are made. Thus,

$$\begin{aligned} \frac{d}{dv} \left(\int_{\varphi_0}^{\varphi_1} \frac{s^2 ds}{\sqrt{U(s)}} \right) &= \frac{\dot{\varphi}_1 - \dot{\varphi}_0}{\varphi_1 - \varphi_0} \int_{\varphi_0}^{\varphi_1} \frac{s^2 ds}{\sqrt{U(s)}} \\ &+ \int_{\varphi_0}^{\varphi_1} \frac{(\dot{\varphi}_1 - \dot{\varphi}_0)s + \dot{\varphi}_0 \varphi_1 - \dot{\varphi}_1 \varphi_0}{\varphi_1 - \varphi_0} \frac{2s ds}{\sqrt{U(s)}} \\ &- \int_{\varphi_0}^{\varphi_1} \left(\dot{U}(s) + \dot{\varphi}_0 U_{\varphi_0}(s) + \frac{(\dot{\varphi}_1 - \dot{\varphi}_0)s + \dot{\varphi}_0 \varphi_1 - \dot{\varphi}_1 \varphi_0}{\varphi_1 - \varphi_0} U'(s) \right) \frac{s^2 ds}{2U^{3/2}(s)}. \end{aligned}$$

As a result, one obtains

$$\ddot{d}(v) = \int_{\varphi_0}^{\varphi_1} \frac{V(s) ds}{2U^{3/2}(s)}, \quad (7.4)$$

with

$$\begin{aligned} V(s) &= 2 \frac{\dot{\varphi}_1 - \dot{\varphi}_0}{\varphi_1 - \varphi_0} s^2 U(s) + (U(s) - s^2)(\dot{U}(s) + \dot{\varphi}_0 U_{\varphi_0}(s)) \\ &+ (4sU(s) - s^2 U'(s)) \frac{(\dot{\varphi}_1 - \dot{\varphi}_0)s + \dot{\varphi}_0 \varphi_1 - \dot{\varphi}_1 \varphi_0}{\varphi_1 - \varphi_0}. \end{aligned} \quad (7.5)$$

From (3.5), (7.1) and (7.2), for the derivatives with respect to v we get respectively

$$\begin{aligned} \dot{\varphi}_0 &= \frac{X_0}{a'(X_0) - v} - \dot{\varepsilon}, \\ \dot{\varphi}_1 &= \frac{X_0 + (p + 3p^2 X_0 + qX_0)\varepsilon^2 + O(\varepsilon^3)}{a'(X_0) - v} + [1 + 2p\varepsilon + 3p^2\varepsilon^2 + O(\varepsilon^3)]\dot{\varepsilon}. \end{aligned} \quad (7.6)$$

Replacing (7.2) and (7.6) in the expression of $V(X)$ given by (7.5), one obtains

$$\begin{aligned} V(X) &= \frac{(1 - s^2)X_0^2}{v - b(X_0)} \left(5 + (3p - m)X_0 + \frac{a'(X_0) - v}{v - b(X_0)} \right) \varepsilon^2 + O(\varepsilon^3) \\ &+ \frac{(1 - s^2)(a'(X_0) - v)\varepsilon^2 \dot{\varepsilon}}{v - b(X_0)} \left[(pX_0^2 + 4X_0 - mX_0^2)s \right. \\ &+ \left(4s^2 + \frac{2(a'(X_0) - v)}{v - b(X_0)} + X_0[p(4 + 8s - 2s^2) + 2ms^2] \right. \\ &\left. \left. + X_0^2[p^2(3 + 2s) + pm(2s^2 - 2s - 1) + \frac{q}{2}(1 + s^2) - (2m^2 + n)s^2] \right) \varepsilon + O(\varepsilon^2) \right]. \end{aligned}$$

The leading term of $\dot{\varepsilon}$ can be determined from (7.3) after differentiation with respect to v . One obtains

$$\dot{\varepsilon} \sim -\frac{\dot{T}_0}{2T_2\varepsilon} = \frac{1}{4(a'(X_0) - v)\vartheta\varepsilon} \left(X_0 m - 3pX_0 - 1 - \frac{a'(X_0) - v}{v - b(X_0)} \right).$$

Finally, replacing in the integral (7.4), we derive the formula

$$\ddot{d}(v) = (a'(X_0) - v) \left[\frac{T_0 X_0^2}{(a'(X_0) - v)^2} - \frac{\dot{T}_0^2}{2T_2} \right] + O(\varepsilon). \quad (7.7)$$

Example 1: The BBM equation. Let $a(u) = 2\omega u + \frac{3}{2}u^2$, $b(u) = 0$. Assume for definiteness that $v > 0$. The other case is considered similarly. Using the notation introduced earlier and the expression of X_0 , we obtain

$$\begin{aligned} a'(X_0) - v &= D^{1/2}, \quad p = -\frac{1}{D^{1/2}}, \quad q = m = n = 0, \quad \vartheta = \frac{15}{16D}, \quad T_0 = 2\pi \frac{v^{1/2}}{D^{1/4}}, \\ T_2 &= \frac{15T_0}{16D}, \quad \dot{T}_0 = \frac{T_0(2\omega^2 - \omega v + 3c_1)}{vD}, \quad \ddot{d}(v) = \frac{T_0}{D^{1/2}} \left[X_0^2 - \frac{8(2\omega^2 - \omega v + 3c_1)^2}{15v^2} \right]. \end{aligned}$$

Now, expressing c_1 and X_0 through D , we rewrite the expression in the brackets of $\ddot{d}(v)$ as

$$\begin{aligned} \Lambda &= X_0^2 - \frac{2(D - v^2 + 2\omega v)^2}{15v^2} \\ &= -\frac{2}{15v^2} \left[D + \sqrt{\frac{5}{6}}vD^{1/2} - (1 - \sqrt{\frac{5}{6}})v(v - 2\omega) \right] \times \\ &\quad \times \left[D - \sqrt{\frac{5}{6}}vD^{1/2} - (1 + \sqrt{\frac{5}{6}})v(v - 2\omega) \right]. \end{aligned}$$

Next, denote by λ_1, λ_2 the roots (with respect to $D^{1/2}$) of the first multiplier, and respectively by λ_3, λ_4 the roots of the second multiplier.

- a) If $0 < v \leq \frac{48}{361}(9 + 5\sqrt{\frac{5}{6}})\omega$, then both multipliers are positive and $\Lambda < 0$.
- b) If $\frac{48}{361}(9 + 5\sqrt{\frac{5}{6}})\omega < v \leq 2\omega$, then the first multiplier is positive, $0 \leq \lambda_4 < \lambda_3$ and therefore $\Lambda > 0 \Leftrightarrow \lambda_4 < D^{1/2} < \lambda_3$.
- c) If $v > 2\omega$, then $0 < \lambda_1 < \lambda_3$, $\lambda_2 < 0$, $\lambda_4 < 0$. Therefore $\Lambda > 0 \Leftrightarrow \lambda_1 < D^{1/2} < \lambda_3$.

Taking a second power of the above inequalities, it follows that

$$\ddot{d}(v) > 0 \Leftrightarrow \begin{aligned} &\lambda_4^2 - (v - 2\omega)^2 < 6c_1 < \lambda_3^2 - (v - 2\omega)^2 \text{ in case b),} \\ &\lambda_1^2 - (v - 2\omega)^2 < 6c_1 < \lambda_3^2 - (v - 2\omega)^2 \text{ in case c).} \end{aligned} \quad (7.8)$$

As λ_k are homogeneous of first degree with respect to v, ω , one can reformulate (7.8) as: $\ddot{d}(v) > 0$ if and only if $(\omega/v, c_1/v^2) \in \Omega \subset \mathbb{R}^2$ where Ω can be explicitly written down if needed by using the formulas of the quadratic roots λ_k . We are not going to do this.

Example 2: The modified BBM equation. Let $a(u) = 2\omega u + \beta u^3$, $b(u) = 0$, $c_1 = 0$.

(i) $X_0 = 0$ (Global center or truncated pendulum case). Then $\ddot{d}(v) = \frac{4\omega^2}{3\beta v^2}T_0 + O(\varepsilon)$ and $\vartheta = \frac{3\beta}{8(v-2\omega)} \neq 0$.

(ii) $X_0^2 = \frac{v-2\omega}{\beta}$ (Duffing oscillator). Then $\ddot{d}(v) = \frac{3v^2-8\omega^2}{6\beta v^2}T_0 + O(\varepsilon)$ and $\vartheta = \frac{3\beta}{4(v-2\omega)} > 0$.

As seen from these examples, $\ddot{d}(v)$ could be negative or positive, depending on the case.

Proof of Theorems IV and V. In Examples 1 and 2 above, we calculated the first term of $\ddot{d}(v)$ provided that $b(u) = 0$ and understood when it is positive. If one take $b(u) = \gamma g(u)$ with $|\gamma|$ small, then $\ddot{d}(v)$ will differ from the case $b(u) = 0$ by a term which is $O(\gamma)$. Therefore $\ddot{d}(v)$ will keep the sign of its first term provided that $|\gamma|$ and the amplitude ε are small enough. This means that $\ddot{d}(v) > 0$ in the domain Ω from the first example and for $3v^2 - 8\omega^2 > 0$ in the Duffing oscillator case of the second example (recall that all solutions not oscillating around zero belong to this case). Since the operator \mathcal{H}_v^0 related to the above cases satisfies the spectral properties required in Assumption 1, the statements of Theorems IV and V follow from Theorems 2 and 3. \square

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